Lecture Eight

Math 101

July 18, 2019

Last time, we saw how certain physical questions naturally give rise to Riemann sums, and hence to definite integrals. Specifically, if \( F(x) \) is the force exerted upon an object at point \( x \) over an interval \([a, b]\), then

\[
W \approx \sum_{i=1}^{N} F(x_i^*) \Delta x_i \xrightarrow{\scriptstyle N \to \infty} \int_a^b F(x) \, dx.
\]

Therefore, the difficulty of solving a work problem lies in setting up the Riemann sum.

Today, we will look at some related constructions. First of all, we define the **average value** of a function \( f(x) \) over an interval \([a, b]\). If we wish to take the arithmetic mean of a finite list of certain values of \( f(x) \), say

\[
\{f(x_1^*), f(x_2^*), \ldots, f(x_n^*)\},
\]

our formula is

\[
f_{av} = \frac{1}{n} (f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)) = \frac{1}{(b-a) \Delta x} \sum_{i=1}^{n} f(x_i^*)
\]

\[
= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x \xrightarrow{\scriptstyle N \to \infty} \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

That is: the average value of \( f(x) \) on \([a, b]\) is defined to be

\[
f_{av} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]
Example 46: Find the average value of $\sin x$ on the interval $[0, \pi]$.

Solution: We use the definition to compute

$$f_{av} = \frac{1}{\pi - 0} \int_0^\pi \sin x \, dx = \frac{1}{\pi} \int_0^\pi \sin x \, dx = \frac{1}{\pi} [-\cos x]_x^\pi = \frac{2}{\pi},$$

and we are done.

Theorem 5.1 (The mean value theorem for integrals). Suppose $f(x)$ is continuous on $[a, b]$. Then there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x) \, dx.$$

That is: $f(x)$ always attains its average value.

Proof. Let $F(x) = \int_a^x f(t) \, dt$. By the first fundamental theorem of calculus, we know that $F(x)$ is differentiable on $(a, b)$ and is continuous on $[a, b]$. Therefore, by the mean value theorem (for derivatives), there exists $c \in (a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a} \left( \int_a^b f(t) \, dt - \int_a^c f(t) \, dt \right) = \frac{1}{b - a} \int_a^b f(x) \, dx.$$

The same theorem implies that $F'(c) = f(c)$, and we are done. 

In fact, in our example, there are two points in the interval in which we have

$$\sin c_1 = \sin c_2 = \frac{2}{\pi} = f_{av}$$

(see left). The theorem doesn’t tell us this, however; it can’t even tell us where these points are.

Related to the concept of the average value is that of the centre of mass, which we now introduce.

It is a fact that, given two point masses $m_1$ and $m_2$ on a lever at respective distances $d_1$ and $d_2$ from the fulcrum, then the system is in equilibrium/fulcrum if and only if $m_1d_1 = m_2d_2$ (see the picture on the next page). In this case, the fulcrum is the centre of mass.
If we draw this picture on the $x$-axis with the centre of mass at $x = \bar{x}$, then we have the equation

$$m_1 (\bar{x} - x_1) = m_2 (x_2 - \bar{x})$$

$$\iff m_1 x_1 + m_2 x_2 = \bar{x} (m_1 + m_2),$$

where mass $m_i$ is at position $x_i$ (for $i = 1, 2$). Thus: the **centre of mass** of two objects is defined

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

By analogy, we define the centre of mass of any finite number of masses $m_1, \ldots, m_n$ at respective positions $x_1, \ldots, x_n$ to be

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

The quantity $\sum_{i=1}^{n} m_i x_i$ in the numerator is the sum of the individual **moments** $m_i x_i$ (with respect to the origin) of the masses $m_i$; their sum is called the **moment of the system** (about the origin).

We now generalize again, to two dimensions. This time, we have masses $m_1, \ldots, m_n$ at respective co-ordinates $(x_1, y_1), \ldots, (x_n, y_n)$, and we define two quantities: the **moment of the system about the $y$-axis**, defined

$$M_y = \sum_{i=1}^{n} m_i x_i,$$

and the **moment of the system about the $x$-axis**, defined

$$M_x = \sum_{i=1}^{n} m_i y_i.$$

Notice the subscripts – **these are not typos!**

We then define the centre of mass $(\bar{x}, \bar{y})$ to be

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m},$$
where
\[ m = \sum_{i=1}^{n} m_i \]
is the total mass of the system.

**Example 47:** Find the centre of mass of the system consisting of three objects at positions \((-1, 1), (2, -1), \) and \((3, 2),\) of respective masses 3, 4, and 8.

**Solution:** Putting these values in terms of our variables gives us
\[ m_1 = 3, (x_1, y_1) = (-1, 1); \quad m_2 = 4, (x_2, y_2) = (2, -1); \quad m_3 = 8, (x_3, y_3) = (3, 2). \]
Thus
\[ m = m_1 + m_2 + m_3 = 3 + 4 + 8 = 15, \]
and similarly
\[ M_y = m_1 x_1 + m_2 x_2 + m_3 x_3 = (3)(-1) + (4)(2) + (8)(3) = 29, \]
and
\[ M_x = m_1 y_1 + m_2 y_2 + m_3 y_3 = (3)(1) + (4)(-1) + (8)(2) = 15. \]
Hence
\[ \bar{x} = \frac{M_y}{m} = \frac{29}{15}, \quad \bar{y} = \frac{M_x}{m} = \frac{15}{15}, \]
and it follows that the centre of mass is at \((\bar{x}, \bar{y}) = \left(\frac{29}{15}, 1\right).\)

These formulas naturally give rise to Riemann sums: if \(R\) is a region in the plane bounded below by \(y = 0,\) above by \(y = f(x),\) to the left by \(x = a,\) and to the right by \(x = b,\) and which moreover has uniform density \(\rho = \rho \text{ kg/m}^2,\) we have for any collection of sample points \((x_i^*, f(x_i^*))\) that
\[ \bar{x} = \frac{M_y}{m} \approx \frac{1}{\rho \int_a^b f(x) \, dx} \sum_{i=1}^{N} \left[ \rho f(x_i^*) \Delta x_i \right] x_i^* \xrightarrow{N \to \infty} \frac{1}{A} \int_a^b xf(x) \, dx, \]
where
\[ A = \int_a^b f(x) \, dx \]
is the area under the graph of \(f(x)\) between \(x = a\) and \(b.\) Similarly,
\[ \bar{y} = \frac{M_x}{m} \approx \frac{1}{\rho \int_a^b f(x) \, dx} \sum_{i=1}^{N} \left[ \rho f(x_i^*) \Delta x_i \right] \left(\frac{1}{2} f(x_i^*)^2\right) \xrightarrow{N \to \infty} \frac{1}{A} \int_a^b \frac{f(x)^2}{2} \, dx; \]
the factor $\frac{1}{2}$ appears because the centre of mass of a uniform rectangle is “halfway up.”

A region $R$ like the one I have just described is called a **lamina**, and its centre of mass – namely,

$$(\bar{x}, \bar{y}) = \left(\frac{1}{A} \int_{a}^{b} x f(x) \, dx, \frac{1}{2} \left(\frac{1}{A} \int_{a}^{b} f(x)^2 \, dx\right)\right)$$

is called its **centroid**.

**Example 48:** Find the centroid of a semicircular disc of radius $R$ and uniform density $\rho$.

**Solution:** We consider the function $f(x) = \sqrt{R^2 - x^2}$, whose graph is the desired lamina. Since our lamina is uniformly dense, we have by our formulas above that

$$\bar{x} = \frac{1}{A} \int_{-R}^{R} x \sqrt{R^2 - x^2} \, dx$$

and

$$\bar{y} = \frac{1}{A} \int_{-R}^{R} \frac{1}{2} \left(\sqrt{R^2 - x^2}\right)^2 \, dx.$$

We know a priori that $A = \frac{1}{2} \pi R^2$; moreover, by symmetry (or using the fact that the integrand is odd), we know that $\bar{x} = 0$, and it remains only to calculate $\bar{y}$. We have

$$\bar{y} = \frac{2}{\pi R^2} \int_{-R}^{R} \frac{1}{2} \left(R^2 - x^2\right) \, dx = \frac{1}{\pi R^2} \int_{-R}^{R} \left(R^2 - x^2\right) \, dx.$$

We can now use the fact that this integrand is even to write

$$\bar{y} = \frac{2}{\pi R^2} \int_{0}^{R} \left(R^2 - x^2\right) \, dx = \frac{2}{\pi R^2} \left[R^2 x - \frac{1}{3} x^3\right]_{x=0}^{R} = \frac{4R}{3\pi},$$

from which we see that our centroid is at $\left(0, \frac{4R}{3\pi}\right)$, regardless of the value of $\rho$. ♣

The other situation we consider is when the region $R$ in question is bounded above and below by curves. In this case, the formula is mostly the same, but slightly different.
Let $R$ be the region bounded above by $y = f(x)$ and below by $y = g(x)$, to the left by the line $x = a$ and to the right by the line $x = b$. Then the centroid of $R$ is given by the formula

$$\bar{x}, \bar{y} = \left( \frac{1}{A} \int_a^b x[f(x) - g(x)] \, dx, \frac{1}{A} \int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2] \, dx \right)$$

where $A$ is as before (we omit the derivation of these formulas here).

**Example 49:** Find the centre of mass of the region bounded by the line $y = x$ and the parabola $y = x^2$.

**Solution:** First, we find the points of intersection to obtain the limits of integration; in example 42 in the last lecture, we found these points to be at $x = 0$ and $x = 1$.

When $x$ is between 0 and 1, then it is a fact that $x^2 \leq x$ (check this!), and we can now use the formulas we just introduced: we have

$$A = \int_0^1 (x - x^2) \, dx = \left[ \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{6},$$

and thus

$$\bar{x} = 6 \int_0^1 x(x - x^2) \, dx = 6 \int_0^1 x^2 - x^3 \, dx = 6 \left[ \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = 6 \left( \frac{1}{12} \right) = \frac{1}{2},$$

$$\bar{y} = 6 \int_0^1 \frac{1}{2}((x)^2 - (x^2)^2) \, dx = 3 \left[ \frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 = \frac{2}{5}.$$

Therefore, the centroid of our region is $(\bar{x}, \bar{y}) = \left( \frac{1}{2}, \frac{2}{5} \right)$.

We leave for now the discussion of “practical calculations” and “real-world scenarios” to introduce the most powerful tool we have in our integral calculus utility belt: *integration by parts*.

### 6 Integration by parts and trigonometric integrals

Recall from the differential calculus the **product rule**, which states that

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$
We can rewrite this equation as

\[ f(x) \frac{d}{dx} g(x) = \frac{d}{dx} (f(x)g(x)) - \left( \frac{d}{dx} f(x) \right) g(x), \]

and by again forgetting that \( \frac{d}{dx} \) is DEFINITELY NOT A FRACTION, we “clear denominators” to get

\[ f(x) d(g(x)) = d(f(x)g(x)) - g(x)d(f(x)). \]

More compactly, with \( u = f(x), v = g(x) \), we have

\[ u \, dv = d(uv) - v \, du; \]

finally, we integrate, and we obtain the integration by parts formula:

\[ \int u \, dv = uv - \int v \, du. \]

Much like \( u \)-substitution, integration by parts requires critical thought only for one single step, and is otherwise totally mechanical.

**Example 50:** Find \( \int x \sin x \, dx \).

**Solution:** We have to make a choice of what is to be \( u \). Just like with \( u \)-substitution, we have to guess, and this is the only difficult part. Once again, once we choose \( u \), the rest of the integrand must be \( dv \) (it’s been decided for us!) So, if we choose \( u = x \), then \( dv \) must be \( \sin x \, dx \).

**Note the inclusion of \( dx \) in the expression for \( dv \)!**

Because \( u = x \), we know that \( \frac{du}{dx} = 1 \), and so \( du = dx \). Similarly,

\[ v = \int dv = \int \sin x \, dx = -\cos x \]

we will save our added constants until the last step. Thus, by our formula,

\[ \int x \sin x \, dx = x \cos x - \int -\cos x \, dx = -x \cos x + \sin x + C. \]

Because this is our final answer, we now include the \( +C \) term. ✾

Some remarks on this example:
1. As with $u$-substitution, making the right choice for $u$ is crucial to make the technique useful.

2. Although we “pick up” an added constant when we integrate $dv$, we will never include it in the formula for $v$ (that is, we will take the added constant to be zero.) We will, however, still need the added constant when we integrate for the last time.

3. For a “geometric” interpretation of integration by parts, see https://en.wikipedia.org/wiki/Integration_by_parts#Visualization.