Last time, we saw with the fundamental theorem of calculus the relationship between integration and differentiation. Specifically, we saw that to compute a definite integral, it is enough to find an antiderivative.

**Definition:** By the expression $\int f(x) \, dx$, called the **indefinite integral** of $f(x)$, we will denote an/the antiderivative of $f(x)$, depending on context. For instance,

$$\int x \, dx = \frac{1}{2} x^2,$$

“kind of”; as we have already seen, any function of the form $\frac{1}{2} x^2 + C$, where $C \in \mathbb{R}$ is constant, is also an antiderivative of $f(x) = x$.

Therefore, we establish the following convention, which we will use throughout the course:

$$\int x \, dx = \frac{1}{2} x^2 + C.$$ 

By this expression, we represent the **entire family** of antiderivatives of $f(x) = x$. Observe that, if we are computing a definite integral, it doesn’t matter which member of this family we pick when we apply FTC II: for instance,

$$\int_a^b x \, dx = \left[ \frac{1}{2} x^2 + C \right]_a^b = \left( \frac{1}{2} b^2 + C \right) - \left( \frac{1}{2} a^2 + C \right) = \left[ \frac{1}{2} x^2 \right]_a^b.$$

It is **crucially important** to distinguish between the real number $\int_a^b f(x) \, dx$, and the function (or family of functions) $\int f(x) \, dx$; the former has **limits of integration** $a$ and $b$, while the latter does not.
Example 24: Suppose \( f(x) = \frac{1}{1+x^2} \). Since
\[
\frac{d}{dx} \arctan x = \frac{1}{1+x^2} = f(x),
\]
we know that \( \int \frac{dx}{1+x^2} \) is the function \( \arctan x + C \). By FTC II, this implies that
\[
\int_0^1 \frac{dy}{1+y^2} = [\arctan y]_y=0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4},
\]
a real number.

Aside: One immediate corollary of FTC, which is inexplicably given its own section in ET[7], is the identity
\[
\int_a^b F'(x) \, dx = F(b) - F(a),
\]
when \( F(x) \) is continuous on \([a,b]\) and differentiable on \((a,b)\).

4 \( u \)-substitution, or: the chain rule in reverse

Suppose I want to integrate \( 2xe^{x^2} \); it is not obvious just by looking at it what an antiderivative might be. However, if we remember the chain rule, we can say that
\[
\int 2xe^{x^2} \, dx = \int \frac{d}{dx} \left( e^{x^2} \right) \, dx,
\]
which by FTC equals
\[
e^{x^2} + C
\]
(don’t forget about the \( C! \)) Here, we introduce a technique that makes this kind of integration trick easier to spot – namely, the \( u \)-substitution.

We follow the same example: first, we’re given a freekie integrand like \( 2xe^{x^2} \), whose antiderivative we don’t know \( a \) priori. Then, we introduce a new variable \( u \), and we try to write the original integral solely in terms of the new variable \( u \).

How do we know what to make \( u \)? Practice!

For our first guess, we’ll take \( u = e^{x^2} \), so that \( \frac{du}{dx} = 2xe^{x^2} \), by the chain rule. Forgetting for the moment that \( \frac{du}{dx} \) is MOST DEFINITELY NOT A FRACTION, we can “multiply by \( dx \)” to get the equation
\[
du = 2xe^{x^2} \, dx,
\]
or even 

\[ \frac{dx}{2xe^{x^2}} = \frac{du}{2xu}. \]

Next, we will look at our original integral, and replace \( e^{x^2} \) by \( u \) wherever we see it:

\[ \int 2xe^{x^2} \, dx = \int 2xu \, dx. \]

Then, we replace \( dx \) by \( \frac{du}{2xu} \) to get

\[ \int 2xe^{x^2} \, dx = \int 2xu \left( \frac{du}{2xu} \right) = \int du, \]

and so

\[ \int 2xe^{x^2} \, dx = \int du = u + C. \]

Putting back in terms of \( x \) completes the job:

\[ \int 2xe^{x^2} \, dx = e^{x^2} + C. \]

Make sense? Of course not! Now let’s do some examples.

[Student]: I’m afraid I don’t understand the method of characteristics.

von Neumann

Young man, in mathematics you don’t understand things. You just get used to them.

**Example 25:** Find \( \int 4x(1 + x^2) \, dx \).

**Solution:** We try the substitution \( u = 1 + x^2 \). Then

\[ \frac{du}{dx} = 2x \iff du = 2x \, dx \iff dx = \frac{du}{2x}. \]

Now we can write

\[ \int 4x(1 + x^2) \, dx = \int 4xu \, dx = \int 4x \cdot u \cdot \left( \frac{du}{2x} \right) = \int 2u \, du. \]

We know by FTC II that \( \int 2u \, du = u^2 + C \), and putting back in terms of \( x \), we get

\[ \int 4x(1 + x^2) \, dx = (1 + x^2)^2 + C \]

as our final answer.
Example 26: Find $\int e^y \cos(e^y) \, dy$.

Solution: Let’s try the substitution $v = e^y$, so that

$$\frac{dv}{dy} = e^y = v \quad \text{and} \quad dy = \frac{dv}{v}.$$ 

Then

$$\int e^y \cos(e^y) \, dy = \int v \cos v \left( \frac{dv}{v} \right) = \int \cos v \, dv = \sin v + C.$$ 

Therefore,

$$\int e^y \cos(e^y) \, dy = \sin(e^y) + C,$$

once we put things back in terms of $y$.

Notice that our ability to take advantage of the substitution depends entirely on making the right choice of what to substitute!

For instance, if in example 26, we instead substitute

$$u = \cos(e^y), \quad \frac{du}{dy} = -\sin(e^y)e^y,$$

then we get

$$\int e^y \cos(e^y) \, dy = \int e^y u \left( \frac{du}{-\sin(e^y)e^y} \right) = -\int \cot(e^y) \, du,$$

which is ⊗, because we still have a $y$ in the expression.

Important fact: $u$-substitution doesn’t work unless we can replace all of the old variables in our integral with the new variables – in this case, we want no $y$s, and only us, to appear in our expression once we’ve made the substitution.

In this case, it’s back to the drawing board. The only real guiding principle is that the quantity we decide to call $u$ should be the “inside function” of the chain rule we’re trying to undo. In example 25, we undid the differentiation

$$\frac{d}{dx} (1 + x^2)^2,$$

and $1 + x^2$ was our choice for $u$; in example 26, we undid

$$\frac{d}{dy} \sin(e^y),$$

and our choice for $v$ was $e^y$. 

4
Example 27: Find \( \int \frac{\cos t}{1 + \sin^2 t} \, dt \).

Solution: Here, we’ll take \( u = \sin t \), so that \( \frac{du}{dt} = \cos t \). Then
\[
\int \frac{\cos t}{1 + \sin^2 t} \, dt = \int \frac{\cos t}{1 + u^2} \, (\frac{du}{\cos t}) = \int \frac{du}{1 + u^2} = \arctan u + C,
\]
or \( \arctan(\sin t) + C \).

Additional exercises: Find the indefinite integrals:

1. \( \int \frac{du}{u \log u} \);
2. \( \int 2 \tan x \sec^2 x \, dx \);
3. \( \int 2 \sec^2 t \tan t \, dt \);
4. \( \int \frac{2y}{\sqrt{1 - y^4}} \, dy \).

(Try to use different substitutions for 2. and 3.)

The short version is this: if \( u = g(x) \) is a differentiable function whose range lies in \([a, b]\), and \( f(x) \) is continuous on \([a, b]\), then
\[
\int f'(g(x))g'(x) \, dx = \int f'(u) \, du = f(u) + C.
\]

We saw many examples of this with indefinite integrals; how does it work with definite integrals?

Clearly, we can do the following process:

1. Take a definite integral \( \int_a^b f(x) \, dx \);
2. Find an antiderivative \( \int f(x) \, dx \) of \( f(x) \) (using \( u \)-substitution);
3. Write the antiderivative in terms of the original variable; and then
4. Evaluate at the limits of integration.

I like to call this the “painful method,” or more broadly, “I like doing unnecessary work.”
Example 28: Find \( \int_1^e \frac{dx}{x(1 + \log^2 x)} \).

Clarification: Like trigonometric functions, log can be exponentiated to denote a power of the function: that is, \( \log^n x \) is defined to be \((\log x)^n\), not \(\log(x^n)\) or \(\log(\log(\cdots(\log x)
\cdots)))\) with \(n\) copies.

Solution: We will substitute \( u = \log x \), so that \( \frac{du}{dx} = \frac{1}{x} \) and \( dx = x \, du \). Then

\[
\int \frac{dx}{x(1 + \log^2 x)} = \int \frac{1}{x(1 + u^2)} \, (x \, du) = \int \frac{du}{1 + u^2} = \arctan u + C;
\]

and so the antiderivative of \( \frac{1}{x(1 + \log^2 x)} \) is \( \arctan(\log x) + C \). By FTC II, therefore, we have

\[
\int_1^e \frac{dx}{x(1 + \log^2 x)} = [\arctan(\log x)]_1^e = \arctan(\log e) - \arctan(\log 1) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.
\]

We got there, but it was time-consuming.

\( \clubsuit \)

There is a better way!

Instead of doing everything (apart from the antidifferentiation) in the “\(x\)-universe,” we will do everything in the “\(u\)-universe,” in this way: when we substitute \( u \) in the integrand, we will also substitute the limits of integration:

\[
\int_a^b f'((g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f'(u) \, du.
\]

So, for the same example we just ran: if \( u = \log x \), then \( u = 0 \) when \( x = 1 \) and \( u = 1 \) when \( x = e \). So:

\[
\int_{x=1}^{x=e} \frac{dx}{x(1 + \log^2 x)} = \int_{u=0}^{u=1} \frac{x \, du}{x(1 + u^2)} = \int_0^1 \frac{du}{1 + u^2} = [\arctan u]_{u=0}^{1} = \frac{\pi}{4}.
\]

It should be clear which method I prefer.
Warning!/Attention!/注意!/¡Atención!/Achtung!/주의!

If you choose to take the “I like doing unnecessary work” path, be **absolutely sure** you have labelled your integrals correctly! The equation above shows one way of doing this, namely by specifying the variables in the limits:

\[
\int_{x=1}^{x=e} \frac{dx}{x(1 + \log^2 x)} = \int_{u=0}^{u=1} \frac{du}{1 + u^2}.
\]

It is tempting, but **WOEFULLY INCORRECT** to write instead

\[
\int_1^e \frac{dx}{x(1 + \log^2 x)} = \int_1^e \frac{xdu}{x(1 + u^2)},
\]

because these are not the limits of integration for the new variable! As I have warned several 101 classes before: to write down a substitution in the second way is **FACTUALLY INCORRECT**, and **I will not award it correct marks on an exam**. It is as incorrect as saying “The speed limit is 100kph, but I’m American, so I can drive 100mph.”

Pay close attention to the limits of integration in the following examples!!

**Example 29:** Compute \( \int_0^3 2ye^{1+y^2} \, dy \).

**Solution:** We try the substitution \( t = 1 + y^2 \), so that

\[
\frac{dt}{dy} = 2y \text{ and } dy = \frac{dt}{2y}.
\]

We have

\[
t(0) = 1 + 0^2 = 1 \text{ and } t(3) = 1 + 3^2 = 10,
\]

and so by our formula

\[
\int_0^3 2ye^{1+y^2} \, dy = \int_1^{10} 2ye^t \left( \frac{dt}{2y} \right) = \int_1^{10} e^t \, dt = \left[ e^t \right]_{t=1}^{10},
\]

which is \( e^{10} - e = e^9(e-1) \).

**Example 30:** Compute \( \int_0^{5\pi/2} \cos(\sin x) \cos x \, dx \).
Solution: We substitute $u = \sin x$, so that
\[ du = \cos x \, dx. \]

Then
\[ u(0) = 0 \quad \text{and} \quad u \left( \frac{5\pi}{2} \right) = 1, \]
and so
\[ \int_{0}^{5\pi/2} \cos(\sin x) \cos x \, dx = \int_{0}^{1} \cos u \, (du) = [\sin u]_{u=0}^{1} = \sin 1 - \sin 0. \]

So the value of the integral is $\sin 1$ (which cannot be simplified).

Example 31: Compute $\int_{0}^{4} \sqrt{2x + 1} \, dx$.

Solution: We take $u = 2x + 1, \, du = 2 \, dx$. Then
\[ \int_{0}^{4} \sqrt{2x + 1} \, dx = \int_{1}^{9} \sqrt{u} \left( \frac{du}{2} \right) = \frac{1}{2} \int_{1}^{9} u^{1/2} \, du = \frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right]_{u=1}^{9} = \frac{1}{3} (9^{3/2} - 1^{3/2}) = \frac{26}{3}. \]

We didn’t discuss the limits this time, but we did make the appropriate changes.

Example 32: Find $\int_{1}^{e} \frac{\log x}{x} \, dx$.

Solution: Take $w = \log x, \, \frac{dw}{dx} = \frac{1}{x}$. Then
\[ \int_{1}^{e} \log x \, \frac{dx}{x} = \int_{0}^{1} w \, dw = \left[ \frac{1}{2} w^{2} \right]_{w=0}^{1} = \frac{1}{2}, \]
and we are done.