Lecture Four

Math 101

July 8, 2019

Last time, we ended our discussion on the Riemann sum. With its definition established, we now look at some properties of the definite integral.

Facts:

1. The definite integral is **linear**, meaning that: if \( f(x), g(x) \) are integrable functions and \( c_1, c_2 \) are real numbers, then
   \[
   \int_a^b (c_1 f(x) + c_2 g(x)) \, dx = c_1 \int_a^b f(x) \, dx + c_2 \int_a^b g(x) \, dx.
   \]
   The remaining facts are special cases of this fact.

2. Taking \( c_2 = 0, f(x) = 1 \), we have
   \[
   \int_a^b c_1 \, dx = c_1 \int_a^b 1 \, dx = c_1(b - a).
   \]

3. Taking \( c_1 = 1, c_2 = -1 \), we have
   \[
   \int_a^b (f(x) - g(x)) \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx.
   \]

Example 17: Compute \( \int_0^1 (2t + 18t^2 - 36t^3) \, dt \).

Solution: By linearity, we can write
\[
\int_0^1 (2t + 18t^2 - 36t^3) \, dt = 2\int_0^1 t \, dt + 18\int_0^1 t^2 \, dt - 36\int_0^1 t^3 \, dt.
\]
Using Riemann sums (exercise), we can show
\[
\int_0^1 t \, dt = \frac{1}{2}, \quad \int_0^1 t^2 \, dt = \frac{1}{3}, \quad \int_0^1 t^3 \, dt = \frac{1}{4},
\]
and so
\[
\int_0^1 (2t + 18t^2 - 36t^3) \, dt = 2 \left( \frac{1}{2} \right) + 18 \left( \frac{1}{3} \right) - 36 \left( \frac{1}{4} \right) = -2,
\]
and we are done.

Linearity implies (as we saw in this example) that it is enough to know the definite integrals of “elementary functions,” i.e. things like
\[x^n, \quad \sin x, \quad e^x, \quad \log x,\]
and to use the known properties of the definite integral.

Below, we list some other properties of the Riemann integral. Suppose \( f, g \) are functions integrable on the interval \([a, b]\); then:

1. If \( f(x) \geq 0 \) for all \( x \in [a, b] \), then
\[
\int_a^b f(x) \, dx \geq 0.
\]
2. If \( 0 \leq f(x) \leq g(x) \) on \([a, b]\), then
\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
\]
3. One has
\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx;
\]
this is known as the **triangle inequality**.

A few words on estimation suppose \( m, M \in \mathbb{R} \) are two numbers that satisfy
\[m \leq f(x) \leq M \text{ for all } x \in [a, b].\]
Then
\[m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a).
\]

**Exercise:** Prove this.
Example 18: Find an estimate for $\int_3^4 e^{-t^2} \, dt$.

Solution: We know that, if $0 < a < b$, then

$$e^{-a^2} > e^{-b^2};$$

so $f(t)$ is decreasing on the interval $(0, \infty)$. In particular, on the interval $[3, 4]$, the function $f(t) = e^{-t^2}$ is no greater than $e^{-9}$ and no less than $e^{-16}$. That is,

$$e^{-16} \leq f(t) \leq e^{-9} \text{ for } t \in [3, 4],$$

and so by our remarks above

$$e^{-t^2} (4 - 3) = e^{-16} \leq \int_3^4 e^{-t^2} \, dt \leq e^{-9} = e^{-3^2} (4 - 3),$$

and we have our estimate.

This is our crudest, simplest estimate for the Riemann integral. Now, we leave Riemann sums behind, and move on to the real substance of the integral calculus.

3 Antiderivatives and the fundamental theorem of calculus

We start by reviewing some of our results from earlier in the week:

$$\int_0^1 2x \, dx = 1 = (1^2 - 0^2);$$

$$\int_2^3 (2x + 3) \, dx = 8 = (3^2 + 3(3)) - (2^2 + 3(2));$$

$$\int_3^4 (1 - 4t) \, dt = -13 = ((4) - 2(4)^2) - ((3) - 2(3)^2).$$

These equalities are not coincidences. In fact, let us introduce some notation: put

$$[f(x)]_a^b := f(b) - f(a).$$
(Here, the symbol := means that the thing on the left is \emph{defined} to be the thing on the right.) Then we can rewrite our results as

\[
\begin{align*}
\int_0^1 \left( \frac{d}{dx}(x^2) \right) \, dx &= [x^2]_{x=0}^1; \\
\int_2^3 \left( \frac{d}{dx}(x^2 + 3x) \right) \, dx &= [x^2 + 3x]_{x=2}^3; \\
\int_3^4 \left( \frac{d}{dt}(t - 2t^2) \right) \, dt &= [t - 2t^2]_{x=3}^4.
\end{align*}
\]

The fundamental theorem of calculus provides an explanation of the connection between differentiation and integration. If \( f(t) \) is an integrable function, we can define a function \( F(x) \) to be the area under the graph of \( f(t) \) between \( a \) and \( x \), for any fixed real number \( a \) in the domain of \( f \); that is, we define

\[
F(x) := \int_a^x f(t) \, dt.
\]

What can we say about \( F(x) \)?

**Theorem 3.1** (The first fundamental theorem of calculus). Let \( f(t) \) be a continuous function defined on \([a, b]\) and let \( F(x) = \int_a^x f(t) \, dt \) for any \( x \in [a, b] \). Then \( F(x) \) is continuous on \([a, b]\), differentiable on \((a, b)\), and satisfies

\[
F'(x) = f(x).
\]

**Proof.** (sketch) We prove that \( F'(x) \) exists and equals \( f(x) \), from which (most of) the other claims follow. Consider the difference quotient

\[
\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \left( \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt \right) = \frac{1}{h} \int_x^{x+h} f(t) \, dt.
\]

If \( h \) is really, really, REALLY small, then \( \int_x^{x+h} f(t) \, dt \) is basically like \( f(x) \cdot h \). Thus

\[
\lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} (f(x) \cdot h) = f(x),
\]

and we are done. \( \Box \)
The short way to remember it is the equation
\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x). \]
That is, in a way, differentiation “undoes” integration.

**Corollary:** Suppose \( F(x) \) is an antiderivative of \( f(x) \), which is to say that \( F(x) \) satisfies \( F'(x) = f(x) \). Then, if \( G(x) \) is another antiderivative of \( f(x) \), there is some constant \( c \in \mathbb{R} \) such that \( F(x) = G(x) + c \).

**Proof.** Let \( H(x) = F(x) - G(x) \); then
\[ H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0. \]
Since \( H'(x) = 0 \), we know that \( H(x) \) is a constant function. \( \square \)

**Theorem 3.2** (The second fundamental theorem of calculus). Let \( f(x) \) be a continuous function on the interval \([a, b]\), and let \( F(x) \) be any antiderivative. Then
\[ \int_a^b f(x) \, dx = F(b) - F(a). \]

**Proof.** By the first fundamental theorem of calculus, we know that
\[ G(x) := \int_a^x f(t) \, dt \]
is an antiderivative of \( f(x) \). By our corollary, we can write \( G(x) = F(x) + c \) for some constant \( c \in \mathbb{R} \); since
\[ G(a) = \int_a^a f(t) \, dt = 0 = F(a) + c \]
(from a previous exercise), we know that \( c = -F(a) \), and so \( G(x) = F(x) - F(a) \). It follows that
\[ G(b) = \int_a^b f(t) \, dt = F(b) - F(a) = [F(t)]_a^b, \]
as claimed. \( \square \)
Exercises: Using the second fundamental theorem of calculus (FTC II), compute:

1. \[ \int_{0}^{2} (2x - x^2) \, dx; \]
2. \[ \int_{1}^{-1} e^x \, dx; \]
3. \[ \int_{\pi}^{5\pi} \cos t \, dt; \]
4. \[ \int_{0}^{1} \frac{dy}{1 + y^2}. \]

Now, we look at several examples of how to apply the fundamental theorem of calculus.

Example 19: Find \( \frac{d}{dx} \int_{1}^{x} \sqrt{1 + t^4} \, dt. \)

Solution: Because \( \sqrt{1 + t^4} \) is continuous on \([1, x]\), we have by the first fundamental theorem of calculus (FTC I) that

\[
\frac{d}{dx} \int_{1}^{x} \sqrt{1 + t^4} \, dt = \sqrt{1 + x^4},
\]

and we are done.

Example 20: Find \( \frac{d}{dx} \int_{1}^{x^4} \sec y \, dy. \)

Solution: As the upper limit is not \( x \), but rather a function of \( x \), we cannot apply FTC I directly. However, we can apply the chain rule: we write

\[
f(x) = \int_{1}^{x} \sec y \, dy, \quad g(x) = x^4,
\]

so that

\[
\int_{1}^{x^4} \sec y \, dy = f(g(x)).
\]
The chain rule tells us that
\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x);
\]
clearly \( g'(x) = 4x^3 \), and according to FTC I we know that
\[
f'(x) = \frac{d}{dx} \int_1^x \sec y \, dy = \sec x,
\]
and thus
\[
\frac{d}{dx} \int_1^x \sec y \, dy = (\sec(x^4))(4x^3) = 4x^3 \sec(x^4),
\]
and we are done.

These are just about the only sort of calculations that can be done using FTC I.

**Example 21:** Find \( \int_0^3 (2x^3 - 6x^2 + x - 1) \, dx \).

**Solution:** To apply FTC II, we need an antiderivative of the integrand, that is, an antiderivative of
\[
2x^3 - 6x^2 + x - 1.
\]
Because differentiation (like integration) is linear, we know that
\[
\frac{d}{dx} \left( af(x) + bg(x) \right) = af'(x) + bg'(x),
\]
where \( a, b \in \mathbb{R} \) are any constants; in particular, this means that it suffices to find an antiderivative of each summand. We compute:

<table>
<thead>
<tr>
<th>summand</th>
<th>( 2x^3 )</th>
<th>( -6x^2 )</th>
<th>( x )</th>
<th>(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>antiderivative</td>
<td>( \frac{1}{2}x^4 )</td>
<td>(-2x^3)</td>
<td>( \frac{1}{2}x^2)</td>
<td>( -x )</td>
</tr>
</tbody>
</table>

It follows that one antiderivative of our integrand is
\[
\frac{1}{2}x^4 - 2x^3 + \frac{1}{2}x^2 - x.
\]

Thus, by FTC II, we have
\[
\int_0^3 (2x^3 - 6x^2 + x - 1) \, dx = \left[ \frac{1}{2}x^4 - 2x^3 + \frac{1}{2}x^2 - x \right]_0^3 = -12,
\]
and we have our integral.

\[ \blacksquare \]
Example 22: Find \( \int_{-\pi/2}^{\pi/2} \cos y \, dy \).

Solution: One antiderivative of \( \cos y \) is \( \sin y + 6 \); so, by FTC II, we have again

\[
\int_{-\pi/2}^{\pi/2} \cos y \, dy = [\sin y + 6]_{x=-\pi/2}^{\pi/2} = (1 + 6) - (-1 + 6) = 2.
\]

Notice that taking \( \sin y + 6 \), instead of \( \sin y \), for our antiderivative, didn’t actually affect the final answer (check this!)

Example 23: Find \( \int_{1}^{e^2} \frac{dt}{t} \).

Solution: We recall that \( \frac{d}{dt} \log t = \frac{1}{t} \), and so again applying FTC II yields

\[
\int_{1}^{e^2} \frac{dt}{t} = [\log t]_{t=1}^{e^2} = \log(e^2) - \log(1) = 2 - 0 = 2.
\]

We begin to see the pattern of how FTC II is applied.

Non-example: Find \( \int_{-1}^{1} \frac{dx}{x^2} \).

Non-solution: One antiderivative of \( \frac{1}{x^2} \) is \( -\frac{1}{x} \) (check this!), so we apply FTC II to get

\[
\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{x=-1}^{1} = -1 - (1) = -2.
\]

B... but \( \frac{1}{x^2} > 0 \) everywhere, so by our fact from before, the integral should be negative...

What went wrong here?

Because the integrand \( \frac{1}{x^2} \) is not continuous on the interval of integration (it has a discontinuity at \( x = 0 \)), we cannot apply the result of FTC II. In fact, \( \frac{1}{x^2} \) is not even integrable on \([-1, 1]\).

Moral: It is essential to check the “premises” of any theorem before we use it! (The “premises” are the things that have to be true for the theorem to apply.) For instance, in the Pythagorean theorem, the premises are that we have a right triangle with hypotenuse \( c \) and other sides \( a \) and \( b \); from these premises, it follows that \( a^2 + b^2 = c^2 \). If the triangle is not right, or the sides are not labelled so that \( c \) is the hypotenuse, then at best the theorem does not apply, and at worst we end up deducing something false (like in our non-example).