Today, we begin by weaving a rich tapestry of examples of sequences.

**Example 9:** Determine $\lim_{n \to \infty} n^r$, for $r \in \mathbb{R}$.

**Solution:** The value of the limit will depend on $r$. For instance, if $r = 0$, then $n^r = 1$ for all $n \in \mathbb{N}$, and so

$$\lim_{n \to \infty} n^r = \lim_{n \to \infty} 1 = 1.$$  

If $r > 0$, then $f(x) = x^r$ is a positive, continuous function on the interval $[1, \infty)$, which is also increasing (because its derivative is $rx^{r-1} > 0$). By fact 2. from the previous lecture, we have that

$$\lim_{n \to \infty} n^r = \lim_{x \to \infty} x^r = \infty.$$  

For $r < 0$, say $r = -R$, we instead have

$$\lim_{n \to \infty} n^r = \lim_{n \to \infty} n^{-R} = \lim_{n \to \infty} (n^{-1})^R.$$  

We saw last time that $\lim_{n \to \infty} n^{-1} = 0$, and because $f(x) = x^R$ is a continuous function of $x$, we have by fact 1. that

$$\lim_{n \to \infty} n^r = \left(\lim_{n \to \infty} n^{-1}\right)^R = (0)^R = 0.$$  

Thus, we have shown

$$\lim_{n \to \infty} n^r = \begin{cases} 
0 & \text{if } r < 0, \\
1 & \text{if } r = 0, \\
\infty & \text{if } r > 0.
\end{cases}$$
Example 10: Determine whether or not the sequence \( a_n = \frac{n}{\sqrt{10+n}} \) is convergent.

Solution: We divide the numerator and denominator both by the highest power of \( n \) appearing in the expression – in this case, it is the first power (that is, \( n^1 = n \)). We have
\[
a_n = \frac{n}{\sqrt{10+n}} = \frac{1}{(1/n)\sqrt{10+n}} = \frac{1}{\sqrt{(10/n^2) + (1/n)}}.
\]

The denominator tends to zero, and assumes only positive values; it follows immediately that its **reciprocal** (or **multiplicative inverse**) tends to \(+\infty\).

In the last example, we waved our hands a little. Let’s record a result so that we don’t have to do this so much.

**Lemma 1.1.** Let \( (a_n) \) be a sequence of nonzero real numbers.

1. If \( \lim_{n \to \infty} a_n = \pm \infty \), then
\[
\lim_{n \to \infty} \frac{1}{a_n} = 0.
\]

2. If \( \lim_{n \to \infty} a_n = 0 \), and only finitely many terms \( a_n \) are negative, then
\[
\lim_{n \to \infty} \frac{1}{a_n} = +\infty.
\]

If instead, only finitely many terms \( a_n \) are positive, then
\[
\lim_{n \to \infty} \frac{1}{a_n} = -\infty.
\]

Proof. (Idea) If \( x_n > M > 0 \) for all \( n \geq N \geq N(M) \), then
\[
0 < \frac{1}{x_n} < \frac{1}{M}
\]
for all \( n \geq N \).

Example 11: Does \( a_n = \frac{n!}{n^n} \) converge or diverge?

Solution: We have from the definitions that
\[
a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdots n}{n \cdot n \cdots n} = \left( \frac{1}{n} \right) \left( \frac{2}{n} \right) \cdots \left( \frac{n}{n} \right).
\]
Clearly, if $1 \leq k \leq n$, then $0 \leq \frac{k}{n} \leq 1$, and so

$$0 < a_n \leq \frac{1}{n}$$

for all $n$.

Thus, since

$$\lim_{n \to \infty} 0 = \lim_{n \to \infty} \frac{1}{n} = 0,$$

we have by the squeeze theorem that $\lim_{n \to \infty} a_n$ exists and equals zero.

**Example 12:** Fill in the table:

<table>
<thead>
<tr>
<th>$a_n$</th>
<th>monotone?</th>
<th>bounded?</th>
<th>convergent?</th>
<th>limit?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\log n}{n}$</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>0</td>
</tr>
<tr>
<td>$-2^n$</td>
<td>yes (s. decr.)</td>
<td>no</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>$(-1)^n$</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>–</td>
</tr>
<tr>
<td>$4 - \frac{1}{n^2}$</td>
<td>yes (s. incr)</td>
<td>yes</td>
<td>yes</td>
<td>4</td>
</tr>
<tr>
<td>$(-1)^n n!$</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>–</td>
</tr>
</tbody>
</table>

**Remarks:**

1. Every convergent sequence is bounded (why?)

2. Adding, deleting, or changing any *finite* number of terms in a sequence *will not* affect boundedness, convergence, or the value of the sequence’s limit (although it may affect monotonicity).

3. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying $f(n) = a_n$ for every $n \in \mathbb{N}$, then monotonicity can be indicated by the derivative. For instance, if

$$f(x) = \frac{\log x}{x},$$

then

$$f'(x) = \frac{1 - \log x}{x^2}.$$  

We compute $f'(2) > 0$, $f'(3) < 0$, and indeed $f'(x) < 0$ whenever $x \geq 3$. This shows that the sequence $a_n$ is decreasing (strictly) from $n = 3$ onward, but *it is not a decreasing sequence*, as

$$0 = a_1 < a_2 = \frac{\log 2}{2}.$$  

However, if we change $a_1$ to some large enough term (say $a_1 = 10$), then the sequence is now decreasing.
Example 13: Show that
\[ \lim_{n \to \infty} 1 + \frac{(-1)^n}{2^n} = 1. \]

Solution: All limit proofs consist of three steps.
1. Let \( \varepsilon > 0 \): we always start with an arbitrary degree of precision.
2. Let \( N = \cdots \): we find the point in the sequence beyond which all terms are "\( \varepsilon \)-close" to the limiting value, in this case 1. Observe that
   \[ |a_n - 1| = \left| \frac{(-1)^n}{2^n} \right| = \frac{1}{2^n}, \]
   and that
   \[ \frac{1}{2^n} < \varepsilon \text{ if and only if } \frac{1}{\varepsilon} < 2^n, \text{ if and only if } \log(1/\varepsilon) < n \log 2, \]
   if and only if \( n > \frac{\log(1/\varepsilon)}{\log 2} \). Thus, we are led to choose \( N \) to be the least integer greater than \( \frac{\log(1/\varepsilon)}{\log 2} \) — but, of course, any integer greater than this number will do.
3. Then (blah blah blah)\ldots: We put it together. If \( n \geq N \), then
   \[ n > \frac{\log(1/\varepsilon)}{\log 2} \iff \frac{1}{2^n} < \varepsilon \iff |a_n - 1| < \varepsilon. \]
   That is: given \( \varepsilon > 0 \), we have found \( N \) such that \( n \geq N \) implies \( |a_n - 1| < \varepsilon \), and so by the definition we have proven
   \[ \lim_{n \to \infty} a_n = 1, \]
as desired.

This ends (for now) our discussion on sequences; we will return in a few weeks.

2 Riemann sums and definite integrals

Now we turn instead to the topic of integration, which will take up most of our time in this course. We start with an unfortunately unmotivated example.
Given ordinary, regular polygons, we know from kindergarten the formulae for their areas:

radius $r$

area $\pi r^2$

side length $s$

area $s^2$

base $b$, height $h$

area $\frac{1}{2}bh$

base width $W$, top width $w$, height $h$

area $\frac{1}{2}h(w + W)$

What if we needed to find the areas of irregular shapes?

For instance, suppose I need to know the total area of my mostly-rectilinear country estate (see right), to reckon my property taxes; call this (green) area $A$. I can get upper and lower bounds on $A$ by tiling my estate with rectangles of a fixed width, say. The smaller and more numerous the rectangles I use, the closer my estimate will be.

To be precise: let $r_i$ be the area of the $i$th red rectangle I use to tile my estate (see below), making sure that no part of any tile is off my land; I will use the red tiles to get an underestimate on the size of my estate.

Similarly, let $b_i$ denote the area of the $i$th blue rectangle I use to tile my estate, making sure that every part of my estate is covered by a tile; this will give me an overestimate.

My estate is bigger than the red area... ... but smaller than the blue area.
In the picture above, we get the estimate

\[ r_1 + r_2 + r_3 + r_4 + r_5 \leq A \leq b_1 + b_2 + b_3 + b_4 + b_5. \]

I can try again with narrower rectangles:

My estate is bigger than the red area... ... but smaller than the blue area.

This time, I get a better bound:

\[ r_1 + r_2 + \cdots + r_{10} \leq A \leq b_1 + b_2 + \cdots + b_{10}. \]

**Question:** How do I know that this new bound is better, and not worse?

If I could take *infinitely many* rectangles, and make them *infinitely small*, then the two sides (the sum of the \( r_i \) and the sum of the \( b_i \)) would “agree” on the true value of \( A \) (think: squeeze theorem). Such is the idea of a *Riemann sum*: we will approximate the area under the graph of a function using narrow vertical rectangles.

**Aside:** check out [https://en.wikipedia.org/wiki/Riemann_sum#Connection_with_integration](https://en.wikipedia.org/wiki/Riemann_sum#Connection_with_integration) for some informative animations.

**Definition:** A *Riemann sum* is an expression of the form

\[ \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{N} f(x_i^*)\Delta x_i, \]

where:

- \( f(x) \) is a function defined on the interval \([a, b]\);
- \( x_i, i = 0, 1, 2, \ldots, n \) are real numbers in \([a, b]\) satisfying
  \[ a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b; \]
• the $x^*_i, i = 0, 1, \ldots, N$ (called **sample points**) are points chosen to satisfy $x_{i-1} \leq x^*_i \leq x_i$; and

• $\Delta x_i = x_i - x_{i-1}$.

More familiarly, we can write a Riemann sum as

$$\sum_{\text{intervals}} \text{(height)} \times \text{(width)}.$$

Some pictures of Riemann sums, with $N = 4$:

**Left** Riemann sum: $x^*_i = x_{i-1}$

**Right** Riemann sum: $x^*_i = x_i$

**Middle** Riemann sum: $x^*_i = \frac{x_{i-1} + x_i}{2}$

**Most** Riemann sums: $x^*_i = ???$

The sample points have been indicated in each picture, and provide an indication as to the origin of their names.
Technically, Riemann sums are very general, and most will look like the bottom-right example. Luckily, we will almost always assume that our intervals are spaced evenly; that is, that

$$\Delta x = \Delta x_i = x_i - x_{i-1} = \frac{b-a}{N} \text{ for every } i = 1, \ldots, N.$$  

This makes it easy for us to write down a formula for the endpoints of our subintervals:

$$x_i = x_0 + i\Delta x = a + i \left( \frac{b-a}{N} \right).$$

Notice that, in our formula for the definition of a Riemann sum, the quantity

$$f(x_i^*)(x_i - x_{i-1})$$

is negative if and only if $f(x_i^*) < 0$. That is to say, if we believe that Riemann sums give us the area “under a graph,” then we should believe that the area below the $x$-axis is negative.

The true area under the graph of the function $f(x)$ between $a$ and $b$, which we have been calling $A$ until now, is denoted by

$$\int_a^b f(x) \, dx,$$

called the (definite) integral of $f(x)$ from $a$ to $b$, and it can be calculated to any arbitrary degree of precision using a finite Riemann sum, if the function $f(x)$ is “nice” enough.

It may seem like a meaningless exercise to try to solve the “area problem” in the first place; but if we can, then we get a lot of other stuff for free.

**Example 14:** My Bentley’s odometer is broken, and my valet needs to know the distance to my country estate. It takes a half-hour to drive there – how far away is it?

**Solution:** I record my speed $v$ (in kilometres per hour) at five-minute intervals, and record the data in a table:

<table>
<thead>
<tr>
<th>$t$</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$</td>
<td>51</td>
<td>61</td>
<td>55</td>
<td>45</td>
<td>53</td>
<td>50</td>
</tr>
</tbody>
</table>

Since average speed is distance divided by time, we know that distance is the product of time and average speed. So, I make the simplifying assumption that, over each
5-minute interval, my speed doesn’t change. Then, if \( t_i, i = 1, \ldots, 6 \) are my 5-minute intervals in hours (not minutes), and \( v(t_i) \) my speed at that time, I can approximate

\[
d_{\text{total}} = \sum_{i=1}^{6} v(t_i) \times \frac{(t_i - t_{i-1})}{\text{length of interval}} = \frac{1}{12}(51 + 61 + 55 + 45 + 53 + 50) = 26.5.
\]

So, my valet must drive approximately 26.5 km to my estate. Of course, if \( v(t) \) is speed in kilometres per hour at time \( t \) (given in hours), then what I have done is approximate

\[
\int_{0}^{1/2} v(t) \, dt
\]

by a right Riemann sum with \( N = 6 \) subintervals. ♣

**Remark:** In the expression

\[
\int_{a}^{b} f(x) \, dx,
\]

the “\( dx \)” term is there to indicate which variable is the variable of integration. While this only becomes important during the calculus of several variables (outside the scope of this class), it is still a necessary part of the expression. Simply writing

\[
\int_{a}^{b} f(x)
\]

is meaningless.

Summarizing the Riemann sum notation: the points \( x_i \) are the endpoints of our subintervals \([x_{i-1}, x_i]\), and the \( x_i^* \) chosen from the subintervals are our sample points. Finally, because we will need it exactly once in the definition below, we will call a tagged partition any collection of subinterval endpoints on \([a, b]\) that are in strict increasing order; that is, an ascending collection of \( N + 1 \) distinct points

\[
a = x_0 < x_1 < \cdots < x_N = b.
\]

**Definition:** The Riemann integral of a function \( f(x) \) on an interval \([a, b]\) exists and equals \( S \) if, for every \( \varepsilon > 0 \), there exists a positive integer \( N \) and a tagged partition \( x_0 < \cdots < x_N \) of \([a, b]\) such that

\[
\left| S - \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1}) \right| < \varepsilon
\]
for any choice of sample points \( \{x_i^*\} \). If the Riemann integral exists, it is denoted

\[
\int_a^b f(x) \, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1}),
\]

and \( f(x) \) itself is called integrable on \([a, b] \). If \( f(x) \) is integrable, then the number \( S \) is finite, and necessarily equals \( \int_a^b f(x) \, dx \).

The limit above is put in scare quotes because, technically, it is not simply the limit as \( N \) goes to infinity; it is the limit as the number of subintervals \( N \) goes to infinity and the length of the longest subinterval goes to zero. This distinction will be unimportant for us.

**Fact:** Every continuous function is integrable, but not every integrable function is continuous.

**Properties:**

1. For any \( a < b \), one has

\[
\int_a^b 1 \, dx = \int_a^b dx = b - a.
\]

2. For any interval \([a, b]\) on which \( f(x) \) is integrable, one has

\[
\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx;
\]

3. For any \( a, b, c \in \mathbb{R} \) for which all integrals make sense, one has

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

**Remark:** The second property is not so much a consequence of our definition, as it is a hint as to how a “tagged partition” is to be interpreted in reverse (but we don’t need to worry about such philosophical subtleties in this class).

**Exercise:** Use one or more of these properties to prove that

\[
\int_a^a f(x) \, dx = 0 \text{ for every } a.
\]

Because Riemann sums are, at heart, finite sums, we review some useful addition identities before we start computing with them.
\[
\begin{align*}
\bullet \quad & \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \\
\bullet \quad & \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \\
\bullet \quad & \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.
\end{align*}
\]

**Example 15:** Compute \( \int_{0}^{1} 2x \, dx \).

**Solution:** Let \( N > 0 \) and let \( x_i \) be evenly spaced, so that
\[
x_i = x_0 + i\Delta x = 0 + \left( \frac{1-0}{N} \right) i = \frac{i}{N},
\]
and \( \Delta x_i = \frac{1}{N} \). We will take a *right* Riemann sum, so \( x_i^* = x_i \), and our Riemann sum is
\[
\sum_{i=1}^{N} f(x_i^*) \Delta x_i = \sum_{i=1}^{N} 2 \cdot \left( \frac{i}{N} \right) \left( \frac{1}{N} \right) = \sum_{i=1}^{N} \frac{2i}{N^2} = \frac{2}{N^2} \sum_{i=1}^{N} i.
\]
By summation identity number 1, this equals
\[
\frac{2}{N^2} \left( \frac{N(N+1)}{2} \right) = \frac{N^2 + N}{N^2} = 1 + \frac{1}{N}.
\]
Taking the limit as \( N \to \infty \), we obtain
\[
\int_{0}^{1} 2x \, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i^*) \Delta x_i = \lim_{N \to \infty} \left( 1 + \frac{1}{N} \right) = 1,
\]
and we have our answer. \( \diamond \)

**Example 16:** Find \( \int_{2}^{4} g(t) \, dt \), where \( g(t) = \begin{cases} 2t + 3 & \text{if } t < 3, \\ 1 - 4t & \text{if } t \geq 3. \end{cases} \)

**Solution:** We start by using useful property number 3 to write
\[
\int_{2}^{4} g(t) \, dt = \int_{2}^{3} g(t) \, dt + \int_{3}^{4} g(t) \, dt = \int_{2}^{3} (2t + 3) \, dt + \int_{3}^{4} (1 - 4t) \, dt.
\]
Consider first the integral from 2 to 3; taking \(N\) evenly spaced intervals as before, we have
\[
t_i = t_0 + i\Delta t = 2 + \left(\frac{3 - 2}{N}\right) \frac{i}{N} = 2 + \frac{i}{N},
\]
and \(\Delta t_i = \frac{1}{N}\) again. Taking again a right Riemann sum (so \(t_i^* = t_i\)), we get
\[
\sum_{i=1}^{N} g(t_i^*)(t_i - t_{i-1}) = \sum_{i=1}^{N} (2t_i^* + 3) \left(\frac{1}{N}\right) = \sum_{i=1}^{N} \left[2 \left(2 + \frac{i}{N}\right) + 3\right] \left(\frac{1}{N}\right)
\]
\[
= \sum_{i=1}^{N} \left(7 + \frac{2i}{N}\right) \left(\frac{1}{N}\right) = \sum_{i=1}^{N} \left(\frac{7}{N} + \frac{2i}{N^2}\right) = \sum_{i=1}^{N} \frac{7}{N} + \sum_{i=1}^{N} \frac{2i}{N^2}.
\]
Clearly, the first sum equals 7, and the second has
\[
\sum_{i=1}^{N} \frac{2i}{N^2} = \frac{2}{N^2} \sum_{i=1}^{N} i = \frac{2}{N^2} \frac{N(N + 1)}{2} = 1 + \frac{1}{N}.
\]
Taking the limit as \(N \to \infty\), we get
\[
\int_{2}^{3} g(t) \, dt = \lim_{N \to \infty} \sum_{i=1}^{N} g(t_i^*)\Delta t_i = \lim_{N \to \infty} \left(7 + \left(1 + \frac{1}{N}\right)\right) = 8.
\]
Similarly, taking \(t_i^* = t_i = 3 + \frac{i}{n}\), we find (exercise) that
\[
\int_{3}^{4} g(t) \, dt = \lim_{N \to \infty} \left(-11 - \left(2 + \frac{2}{N}\right)\right) = -13,
\]
and hence
\[
\int_{2}^{4} g(t) \, dt = 8 + (-13) = -5.
\]
So \(g(t)\) is, in fact, integrable, even though it is not continuous.

These are all of the examples of Riemann sums we will do! Although they are an essential part of the theoretical framework underpinning the integral calculus, one does not generally compute integrals using the method in the above two examples.