Last time, we introduced sequences, some of their properties, and saw some examples. Today we begin by defining a very important notion; namely, that of the limit.

**Definition:** Let \((a_n)\) be a sequence and let \(L \in \mathbb{R}\); then \(L\) is said to be the limit of the sequence \((a_n)\), denoted 
\[
\lim_{n \to \infty} a_n = L,
\]
if, for every \(\varepsilon > 0\), there exists some \(N \in \mathbb{N}\) such that
\[
|a_n - L| < \varepsilon
\]
whenever \(n \geq N\).

**Note:** For shorthand, one often sees \(a_n \to L\) or \(a_n \xrightarrow{n \to \infty} L\) instead of \(\lim_{n \to \infty} a_n = L\); all mean the same thing.

The definition is very technical, so we’ll break it into parts to try to understand it better.

- **For all** \(\varepsilon > 0\) – to any degree of precision, except exactness (remember: an exact approximation is no longer an approximation.)

- **there exists some** \(N\) – if we know what \(\varepsilon\) is, then we can pick a special term in the sequence.

- **such that ... whenever** \(n \geq N\). – if the point \(a_k\) is more than \(\varepsilon\) away from the value \(L\), then \(a_k\) is one of the first \(N\) terms in the sequence.

**Equivalently:** The limit as \(n\) tends to infinity of \(a_n\) equals \(L\) if and only if, for every \(\varepsilon > 0\), there are only finitely many terms \(a_n\) that are at least \(\varepsilon\) away from \(L\).
If there exists some real number $L$ such that $\lim_{n \to \infty} a_n = L$, then the sequence $(a_n)$ is said to be **convergent**, and we say that the sequence $(a_n)$ **converges to** $L$. Otherwise, $(a_n)$ is said to be **divergent**.

**Example 4:** The sequences

$$a_n = \frac{1}{n}, \quad b_n = 1 + \frac{(-1)^n}{2^n}, \quad c_n = \frac{2n + 7}{3n - 1}$$

are all convergent, with respective limits 0, 1, and $\frac{2}{3}$.

**Example 5:** The sequences

$$d_n = n, \quad e_n = (-1)^n \log n, \quad f_n = e^n$$

are all divergent.

We notice something particular about the sequences $(d_n)$ and $(f_n)$: namely, that their terms become arbitrarily large as $n \to \infty$. This motivates another definition.

**Definition:** The sequence $(a_n)$ is said to **diverge to infinity**, written $\lim_{n \to \infty} a_n = \infty$ (or $+\infty$) if, for all $M > 0$, there exists $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq M$. Equivalently: the sequence $(a_n)$ diverges to infinity if, given any positive number $M$, only finitely many terms $a_n$ satisfy $a_n < M$.

**Exercise 3:** Try to give a definition for the notion of **divergence to** $-\infty$.

**Example 6:** The sequence $a_n = n + \frac{(-1)^n}{n}$ diverges to infinity, as can be seen by the fact that $a_n \geq n - 1$ for all $n \in \mathbb{N}$.

**Example 7:** The sequence $b_n = 1 - \frac{n}{\log n}$ (defined for $n \geq 2$) diverges to $-\infty$; this can be proven directly, but it is more easily seen as a consequence of the following theorem.
Theorem 1.1. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence, and let
\[ f : \mathbb{R}_{>0} \to \mathbb{R} \]
be a real-valued function of positive real numbers, satisfying \(f(n) = a_n\)
for all \(n \in \mathbb{N}\). Then, if
\[ \lim_{x \to \infty} f(x) \]
exists, then so does \(\lim_{n \to \infty} a_n\), and moreover
\[ \lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x). \]

Proof. (Sketch) Suppose \(\lim_{x \to \infty} f(x) = c \in \mathbb{R}\); then (remembering our definition
from differential calculus) this means that, for any \(\varepsilon > 0\), the only values of \(x \in \mathbb{R}\)
for which \(f(x)\) is at least \(\varepsilon\) away from \(c\) lie below some fixed real number, say
\(x_0 = x_0(\varepsilon)\). In particular, every \(a_n = f(n)\) for \(n \geq x_0\) is within \(\varepsilon\) of \(c\). \(\Box\)

Corollary: With the same notation,
\[ \lim_{x \to \infty} f(x) = \pm \infty \text{ implies } \lim_{n \to \infty} a_n = \pm \infty. \]

In both cases (those of convergence, and of divergence to \(\pm \infty\)), the important thing
to notice is not that infinitely many terms of the sequence are arbitrarily close to
the limiting value. The important thing to notice is that only finitely many terms
are not arbitrarily close to the limit.

Example 8: Find \(\lim_{n \to \infty} \frac{1}{n}\), if it exists.

Solution: Our intuition tells us that the limit should be zero; let’s try to prove this
from the definition. We need to find \(N \in \mathbb{N}\) such that
\[ n \geq N \text{ implies } |a_n - 0| < \varepsilon, \text{ i.e. } \left| \frac{1}{n} \right| < \varepsilon. \]

We observe
\[ \frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n; \]
so, if we take \(N\) to be the least integer greater than \(\frac{1}{\varepsilon}\), then
\[ n \geq N \text{ implies } n > \frac{1}{\varepsilon}, \text{ if and only if } \varepsilon > \frac{1}{n} = |a_n - 0|. \]
Thus, given $\varepsilon > 0$, we have found $N = N(\varepsilon)$ such that $n \geq N$ implies $|a_n| < \varepsilon$; it now follows from the definition that

$$\lim_{n \to \infty} a_n = 0.$$ 

Clearly, it is a little cumbersome to deal with the definition. We will often rely on some helpful facts to save ourselves from such busywork.

**Facts:**

1. If $f(x)$ is continuous and $(a_n)$ is convergent, then
   $$\lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right);$$
   we say that the taking of limits **commutes** with continuous functions (it doesn’t matter which one we apply first).
2. If $\lim_{n \to \infty} a_n = \pm \infty$, then
   $$\lim_{n \to \infty} f(a_n) = \lim_{x \to \pm \infty} f(x).$$
3. If $\lim_{n \to \infty} |a_n| = 0$, then
   $$\lim_{n \to \infty} a_n = 0.$$

We will close by listing a few more useful facts and tools, before getting lost next lecture in a sea of examples.

**Properties of convergent sequences:**

Let $(a_n), (b_n)$ be convergent sequences with

$$\lim_{n \to \infty} a_n = L \text{ and } \lim_{n \to \infty} b_n = M.$$ 

Then for any real number $c$ and any positive real number $p$, we have:

$$c a_n = c L \quad \text{and} \quad |a_n| < \varepsilon \implies |b_n| + |c| E < \varepsilon + |c| E.$$ 

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**Warning! Attention! 注意! ¡Atención! Achtung! 주의!**

The converse of the corollary is **not true**!

For instance, the sequence $a_n = n \sin(n\pi)$

is just a fancy way of defining the constant sequence $(0, 0, 0, \ldots)$; by contrast, the function $f(x) = x \sin(x\pi)$ attains arbitrarily large positive and negative values (see below).
1. \( \lim_{n \to \infty} (a_n \pm b_n) = L \pm M; \)
2. \( \lim_{n \to \infty} c \cdot a_n = cL; \)
3. \( \lim_{n \to \infty} a_nb_n = LM; \)
4. if only finitely many terms \( a_n \) are negative, then \( \lim_{n \to \infty} a_n^p = L^p; \) and
5. if \( M \neq 0 \) and all \( b_n \neq 0 \), then \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{M}. \)

(Most of these facts are easy consequences of the first fact).

**Theorem 1.2** (Squeeze theorem). Let \((a_n), (b_n), (c_n)\) be three sequences satisfying
\[ a_n \leq b_n \leq c_n \text{ for all } n \geq N_0, \]
for some \( N_0 \in \mathbb{N} \). If
\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \in \mathbb{R}, \]
then \((b_n)\) is convergent, and
\[ \lim_{n \to \infty} b_n = L \text{ also.} \]

**Theorem 1.3** (Monotone convergence theorem). If a sequence is both bounded and monotone, then it is convergent.

**Corollary 1** (Bolzano-Weierstrass theorem). Every bounded sequence has a convergent subsequence.