Let’s review the convergence tests that we have accumulated.

<table>
<thead>
<tr>
<th>Name</th>
<th>Type of series</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Divergence test</strong></td>
<td>Any</td>
<td>Quick and easy</td>
<td>Only discounts obvious series</td>
</tr>
<tr>
<td><strong>Geometric series test</strong></td>
<td>Geometric</td>
<td>Easy to check</td>
<td>Only applies to one kind of series</td>
</tr>
<tr>
<td>$p$-test</td>
<td>$\sum_{n=1}^{\infty} \frac{1}{n^p}, p \in \mathbb{R}$</td>
<td>Easy to check</td>
<td>Only applies to one kind of series</td>
</tr>
<tr>
<td><strong>Comparison test</strong></td>
<td>Nonnegative series</td>
<td>Intuitive</td>
<td>Often difficult to find comparable series</td>
</tr>
<tr>
<td><strong>Integral test</strong></td>
<td>Nonnegative, eventually decreasing series</td>
<td>An “if and only if” test; reduces the problem to calculus</td>
<td>Very restrictive hypotheses</td>
</tr>
<tr>
<td><strong>Limit comparison test</strong></td>
<td>Positive series</td>
<td>An “if and only if” test</td>
<td>Limit can be difficult to evaluate</td>
</tr>
<tr>
<td><strong>Alternating series test</strong></td>
<td>Decreasing, alternating series</td>
<td>Hypotheses easy to check</td>
<td>Very restrictive hypotheses</td>
</tr>
<tr>
<td><strong>Ratio/root test</strong></td>
<td>Any</td>
<td>Kills negatives; “plug and chug”</td>
<td>Can’t detect conditional convergence</td>
</tr>
</tbody>
</table>

In addition, we must also remember to keep an eye out for telescoping sums, and other tricks.
Returning now to power series: we saw last time how a series in a formal variable $x$ can give us a function of $x$. Today, we turn this construction around, and ask: given a function and a point, is there a power series centered on the given point that equals the given function?

**Example 107:** Express $\frac{1}{1+x^2}$ as a power series, and find its radius and interval of convergence.

**Solution:** We have seen how $\frac{1}{1-x}$ can be written as a power series about zero:

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \text{ whenever } |y| < 1.$$  

If we write $y = x^2$, then $|y| < 1$ if and only if $|x| < 1$; so, for $x \in (-1, 1)$, we can write

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \frac{1}{1-y} = \sum_{n=0}^{\infty} (-y)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$$

We know, therefore, that this power series has radius of convergence at least one; because it diverges when $x = \pm 1$, we know that the radius of convergence is exactly one, and that the interval of convergence is $(-1, 1)$.

In general, *the only way we have* of finding power series is by working backwards from a geometric series.

**Example 108:** Express $\frac{1}{x+2}$ as a power series, and find its radius and interval of convergence.

**Solution:** Let us leave aside for the moment questions of convergence, and instead see what “formal manipulation” of the series gives us. We have

$$\frac{1}{2+x} = \frac{1}{2} \left( \frac{1}{1 + \left( \frac{x}{2} \right)} \right) = \frac{1}{2} \left( \frac{1}{1 - \left( -\frac{x}{2} \right)} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \left( -\frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}},$$

which will be valid whenever the series converges. The geometric series $\frac{1}{1-(-x/2)}$ will converge exactly when $\left| -\frac{x}{2} \right| < 1$, which is exactly when $|x| < 2$, so again the radius
of convergence is at least 2. To see that it is exactly two, we need only show that the
series diverges at \( x = \pm 2 \); indeed,

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(\pm 2)^n}{2n+1} = \frac{1}{2} \sum_{n=0}^{\infty} (\mp 1)^n,
\]

which diverges. It follows that the interval of convergence is \((-2, 2)\), and that the
radius of convergence is 1.

We like power series because they behave well analytically (i.e. calculus-wise) on their
intervals of convergence.

**Theorem 11.2.** Let \( \sum_{n=0}^{\infty} a_n(x - c)^n \) be a power series with radius of
convergence \( R > 0 \) or \( R = \infty \), and define

\[
f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n
\]

for \( x \in (c - R, c + R) \). Then:

1. \( f(x) \) is continuous and differentiable on \((c - R, c + R)\).

2. \( f'(x) \) has power series \( f'(x) = \sum_{n=0}^{\infty} na_n(x - c)^{n-1} \) about \( c \).

3. \( \int f(x) \, dx \) has power series \( \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + K \) about \( c \), where
   \( K \) is the added constant of integration.

Moreover, the radius of convergence of the series in cases 2. and 3. is
also \( R \).

**Warning!/Attention!/注意!/¡Atención!/Achtung!/주의!**

Just because the radius of convergence stays the same, it doesn’t mean that the
interval of convergence stays the same (unless \( R = \infty \)). In general, to determine the
interval of convergence, we will still need to check the endpoints separately.

This theorem gives us a lot of power. For instance, because we know

\[
\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{1}{(1-x)^2} \quad \text{and} \quad \int \frac{dx}{1-x} = -\log |1-x| + C,
\]
we immediately deduce the power series expansions

\[
\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \quad x \in (-1, 1),
\]

and

\[
\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right), \quad x \in (-1, 1).
\]

(Of course, any power series for the antiderivative of \(\frac{1}{1-x}\) must include an added constant of integration.)

**Example 109:** Find a power series expansion for \(f(x) = \arctan x\).

**Solution:** Rather than look at \(f(x)\) directly, we consider its derivative, namely

\[
f'(x) = \frac{1}{1+x^2}.
\]

We saw in example 107 that

\[
f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}
\]

is a valid power series expansion for \(x \in (-1, 1)\); so, by theorem 11.2, we also have for \(x \in (-1, 1)\) that

\[
f(x) = \int \frac{dx}{1+x^2} = K + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1},
\]

for some constant \(K\). Because \(f(0) = \arctan(0) = 0\), we must have that

\[
0 = K + \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+1}}{2n+1} = K,
\]

and so

\[
\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]
on $(-1, 1)$. By the alternating series test (exercise), this series also converges at the endpoints $x = \pm 1$. We deduce that the interval of convergence is $[-1, 1]$, strictly larger than the interval of convergence of its derivative, and we are done.

\textbf{Remark:} Evaluating at $x = 1$ gives the (very slowly converging) \textbf{Leibniz formula} for $\pi$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

\textbf{Fact:} Two power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, $\sum_{n=0}^{\infty} b_n (x-d)^n$, are equal if and only if $c = d$ and $a_n = b_n$ for every $n$. Equivalently: a power series is determined by its centre and its coefficients.

\textbf{Example 110:} Suppose $f(x) = e^x$ admits a power series centered at $x = 0$ with radius of convergence $R > 0$ or $R = \infty$. Find the power series, and its interval and radius of convergence.

\textbf{Solution:} We know that $f'(x) = f(x)$; so, if we write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then theorem 11.2 tells us

$$f'(x) = \sum_{n=0}^{\infty} na_n x^{n-1}.$$ 

That is: whatever the coefficients $a_n$ are, they must satisfy the equality of series

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} na_n x^{n-1}.$$ 

Let us re-index the second series, so that we can compare coefficients more easily: if we write $m = n - 1$, then

$$f'(x) = \sum_{n=0}^{\infty} na_n x^{n-1} = \sum_{m=-1}^{\infty} (m+1)a_{m+1} x^m = \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m$$

(the last equality holds because the term corresponding to $m = -1$ is zero). Thus, because our power series have the same centre (i.e. $x = 0$), we know that they are equal if and only if

$$a_n = (n+1)a_{n+1} \text{ for every } n = 0, 1, 2, \ldots,$$
or equivalently,

\[ a_n = \frac{a_{n-1}}{n} = \frac{a_{n-2}}{n(n-1)} = \cdots = \frac{a_0}{n(n-1) \cdots (2)(1)} = \frac{a_0}{n!}. \]

It follows that every coefficient \( a_n \) is determined by the value of the constant term \( a_0 \); and, because

\[ a_0 = \sum_{n=0}^{\infty} a_n(0)^n = f(0) = e^0 = 1, \]

we have that \( a_n = \frac{1}{n!} \), and so \( e^x \) has a (non-stupid) power series centered on zero, then it is

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]

It remains now only to check where/if the series converges. Fix a real number \( x \): we have

\[ \left| \frac{x^{n+1}}{(n+1)!} \right| = \left| \frac{x^n}{n+1} \right|, \]

which tends to zero as \( n \to \infty \), regardless of \( x \). Because zero is less than one, the ratio test implies that the series converges (absolutely), and because \( x \) was arbitrary we deduce that the series converges for all \( x \in \mathbb{R} \). That is: the radius of convergence is infinite, and the power series of \( f(x) \) is

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \]

valid on \((-\infty, \infty)\).

The observation we made in this example generalizes. Suppose

\[ f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n \]

on some nonempty interval \((a, b)\), so that \( f(c) = a_0 \) in particular. Then, for \( x \in (a, b) \), theorem 11.2 tells us that

\[ f'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1}, \]
so \( f'(c) = a_1; \)
\[
f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2},
\]
so \( f''(c) = 2a_2; \)
\[
f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-c)^{n-3},
\]
so \( f'''(c) = 6a_3. \) In general, we have
\[
f^{(k)}(c) = k!a_k, \text{ so } a_k = \frac{f^{(k)}(c)}{k!},
\]
because
\[
f^{(k)}(c) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}a_n(x-c)^{n-k}.
\]
The following is an immediate consequence of this observation.

**Fact:** If there exists a power series \( \sum_{n=0}^{\infty} a_n(x-c)^n \) with radius of convergence \( R > 0 \) or \( R = \infty \) such that
\[
f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n,
\]
then the coefficients of this power series are
\[
a_n = \frac{f^{(n)}(c)}{n!}.
\]
That is: *any non-stupid power series expansion of \( f(x) \) about \( x = c \) is completely determined by the derivatives of \( f(x) \) at \( c \).*

Let us give a name to these “non-stupid” power series.

**Definition:** Let \( f(x) \) be an infinitely differentiable function. The **Taylor series** of \( f(x) \) about \( x = c \) is defined to be the power series
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots
\]
The **Maclaurin series** is the Taylor series about \( x = 0: \)
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-c)^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots
\]
Remark: We can now restate our fact above as: If \( f(x) \) admits a power series expansion that converges for more than one value of \( x \), then this power series is a Taylor series. Equivalently: If there exists \( R > 0 \) such that

\[
f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n \quad \text{whenever} \quad |x - c| < R,
\]

then \( a_n = \frac{f^{(n)}(c)}{n!} \).

Example 111: In example 110 we found the Maclaurin series of \( f(x) = e^x \).

Example 112: Find the Maclaurin series for \( f(x) = \cos x \) and \( g(x) = \sin x \).

Solution: We first compute derivatives:

\[
\begin{align*}
f'(x) &= -\sin x, \quad g'(x) = \cos x, \\
f''(x) &= -\cos x, \quad g''(x) = -\sin x, \\
f'''(x) &= \sin x, \quad g'''(x) = -\cos x, \\
f(4)(x) &= \cos x = f(x), \quad g(4)(x) = \sin x = g(x).
\end{align*}
\]

Clearly these derivatives will repeat in this cycle of four, so these are all the derivatives we need. We compute

\[
f^{(n)}(0) = \pm \sin(0) = 0 \text{ if } n \text{ is odd}, \quad g^{(m)}(0) = \pm \sin(0) = 0 \text{ if } m \text{ is even},
\]

and similarly

\[
f^{(n)}(0) = (-1)^{n/2} \text{ if } n \text{ is even}, \quad g^{(m)}(0)(-1)^{(m-1)/2} \text{ if } m \text{ is odd}.
\]

So, if

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n x^n,
\]

then

\[
a_n = \begin{cases} 
0 & \text{if } n \text{ is odd,} \\
\frac{(-1)^{n/2}}{n!} & \text{if } n \text{ is even,}
\end{cases} \quad \text{and} \quad b_n = \begin{cases} 
0 & \text{if } n \text{ is even,} \\
\frac{(-1)^{(n-1)/2}}{n!} & \text{if } n \text{ is odd,}
\end{cases}
\]

and so

\[
f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]
and
\[ g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]
The ratio test tells us (exercise) that these series converge for all \( x \in \mathbb{R} \).

Using summation notation, we have just shown that
\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{and} \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.
\]

**Corollary:** If \( i = \sqrt{-1} \), then \( e^{ix} = \cos x + i \sin x \).

**Proof.** We note that
\[
i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1,
\]
and we apply the Maclaurin series we have computed. We have:
\[
e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots
\]
\[
= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)
\]
\[
= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right) + i \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \cos x + i \sin x.
\]

One brief aside about the geometric series: our usual formula
\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad x \in (-1, 1),
\]
is in fact the Maclaurin series for \( f(x) = \frac{1}{1-x} \). What if we wanted a different Taylor series for \( f(x) \), namely one centered at another point?

Let \( a \in (-1, 1) \); then we have
\[
\frac{1}{1-x} = \frac{1}{1-x+a-a} = \frac{1}{(1-a)-(x-a)} = \frac{1}{1-a} \left(\frac{1}{1-\frac{x-a}{1-a}}\right).
\]
If \(|\frac{x-a}{1-a}| < 1\) (which is the case if and only if \(2a - 1 < x < 1\)), then this equals

\[
\frac{1}{1-a} \sum_{n=0}^{\infty} \left(\frac{x-a}{1-a}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(1-a)^{n+1}}(x-a)^n.
\]

Notice that we have actually \textit{changed} the interval of convergence by this process! It used to be \(x \in (-1, 1)\), and now it is \((2a - 1, 1)\). If we take \(a\) to be negative, then we can now \textit{force} the series to converge at \(x = -1\): plugging this into our new power series gives

\[
\frac{1}{1-a} \sum_{n=0}^{\infty} \left(-\frac{1-a}{1-a}\right)^n = \frac{1}{1-a} \left(\frac{1}{1-\frac{1-a}{1-a}}\right) = \frac{1}{1-a} \left(\frac{1-a}{2}\right) = \frac{1}{2},
\]

which is exactly \(\frac{1}{1-(-1)}\). Look at that! It’s CRAZY.

We tabulate the Maclaurin series we have computed so far:

<table>
<thead>
<tr>
<th>Function</th>
<th>Maclaurin series</th>
<th>Radius of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{1-x})</td>
<td>(\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots)</td>
<td>(R = 1)</td>
</tr>
<tr>
<td>(e^x)</td>
<td>(\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)</td>
<td>(R = \infty)</td>
</tr>
<tr>
<td>(\sin x)</td>
<td>(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)</td>
<td>(R = \infty)</td>
</tr>
<tr>
<td>(\cos x)</td>
<td>(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots)</td>
<td>(R = \infty)</td>
</tr>
<tr>
<td>(\arctan x)</td>
<td>(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots)</td>
<td>(R = 1)</td>
</tr>
<tr>
<td>(\log(1 + x))</td>
<td>(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots)</td>
<td>(R = 1)</td>
</tr>
</tbody>
</table>

You will be expected to know these series.

**Example 113:** Find \(\int e^{-x^2} \, dx\).

**Solution:** At long last, we find the anti-derivative of \(e^{-x^2}\)!
Because the power series for $e^y$ converges for all real numbers $y$, we know in particular that it will converge when $y = -x^2$, and so we can substitute this into the power series to obtain a new, convergent power series. We have

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!},$$

which converges for all $x \in \mathbb{R}$; by theorem 11.2, we can integrate term-by-term. We obtain

$$\int e^{-x^2} \, dx = \int \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \right) \, dx = K + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n} \, dx,$$

where $K$ is the constant of integration. For all $n \geq 0$ we know

$$\int x^{2n} \, dx = \frac{x^{2n+1}}{2n + 1},$$

and we have that the antiderivative of $e^{-x^2}$ is given by the power series

$$\int e^{-x^2} \, dx = K + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n + 1)} = K + x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots,$$

for all $x \in \mathbb{R}$.

We can also multiply power series as we would polynomials. When we do this, the interval of convergence of the product will be the intersection of the intervals of convergence of the factors. That is: the product of two power series converges exactly where both of those power series converge.

**Fact:** Let $\sum_{n=0}^{\infty} a_n (x - c)^n$, $\sum_{n=0}^{\infty} b_n (x - c)^n$ be two power series with the same centre.

On its interval of convergence, the power series of the product

$$\left( \sum_{n=0}^{\infty} a_n (x - c)^n \right) \left( \sum_{n=0}^{\infty} b_n (x - c)^n \right)$$

is given by $\sum_{n=0}^{\infty} d_n (x - c)^n$, where

$$d_n = \sum_{i+j=n} a_i b_j.$$
The sum is taken over all nonnegative integers \( i, j \), such that \( i + j = n \). For instance, the coefficient of \((x - c)\) in the product is \(a_0b_1 + a_1b_0\), and the coefficient of \((x - c)^3\) is

\[a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0.\]

**Example 114:** Find the Maclaurin series for \((a) \ f(x) = e^x \sin x\) and \((b) \ g(x) = \frac{e^x}{1-x}\), and their intervals of convergence.

**Solution:** (a) We have, for every \( x \in \mathbb{R} \),

\[e^x \sin x = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)

By our fact above, this product can be written \(\sum_{n=0}^{\infty} d_n x^n\), where:

\[d_0 = (1)(0) = 0,\]
\[d_1 = (1)(1) + (1)(0) = 1,\]
\[d_2 = (1)(0) + (1)(1) + (\frac{1}{2!})(0) = 1,\]
\[d_3 = (1)\left(\frac{-1}{3!}\right) + (1)(0) + (\frac{1}{2!})(1) + (\frac{1}{3!})(0) = \frac{1}{3};\]
\[\vdots\]

It follows that the Maclaurin series for \(e^x \sin x\) is given by

\[x + x^2 + \frac{x^3}{3} + \cdots,\]

and that it converges everywhere.

(b) As before, we write

\[\frac{e^x}{1-x} = (e^x) \left(\frac{1}{1-x}\right) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \left(1 + x + x^2 + x^3 + \cdots\right) = 1 + (1+1)x + (1+1+1)\frac{1}{2!}x^2 + (1+1+1+\frac{1}{3!})x^3 + \cdots = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \cdots\]
(Because the coefficient $b_n$ of $x^n$ is 1 for every $n$ in the second series, the products $a_ib_j$ we calculate are really just $a_i$.) Because the power series for $\frac{1}{1-x}$ diverges outside of $(-1, 1)$, we know that this is the largest open interval on which this power series converges.

In fact, the divergence test implies that this series will diverge for $x = \pm 1$, but this doesn’t follow from our remarks above.

This ends the new material for the course. Next time, we will review.