Lecture Fourteen

Math 101

August 2, 2019

Last time, we closed with a new concept, namely that of absolute or conditional convergence. We also saw that, to show that a series converges, it is enough to show that it converges absolutely — although, of course, not every convergent series is absolutely convergent.

We briefly state the following theorem, which we will not need in this course.

**Theorem 10.9** (Riemann series theorem). If \( \sum_{n=1}^{\infty} a_n \) is conditionally convergent, then for any value \( c \in \mathbb{R} \cup \{\pm \infty\} \), there exists a reordering of the terms \( a_{\sigma(n)} \) (say) such that \( \sum_{n=1}^{\infty} a_{\sigma(n)} = c \).

**Proof.** (Idea) To be conditionally convergent, the series necessarily has infinitely many positive terms and infinitely many negative terms, both of which converge to zero (by the divergence test). With \( c > 0 \) fixed, we add positive terms until the partial sum exceeds \( c \), then negative terms until the partial sum is less than \( c \), and so on.

**Example 98:** Determine whether or not the series \( \sum_{n=1}^{\infty} \frac{\cos n}{n^2} \) is convergent.

**Solution:** The divergence test is unhelpful here, and the comparison, integral, and alternating series test cannot be applied (the series is neither alternating, nor made up of solely positive terms). However, we see that

\[
0 \leq \left| \frac{\cos n}{n^2} \right| = \left| \frac{\cos n}{n^2} \right| \leq \frac{1}{n^2},
\]

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so the series $\sum_{n=1}^{\infty} |a_n|$ converges, by the comparison test. It follows that $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent, and so in particular (by proposition 10.8) is convergent.

We like absolute convergence because, unlike in the situation encountered in the Riemann series theorem, every absolutely convergent series converges to exactly one value, no matter how the terms of the series are permuted. That is, if $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is some permutation/rearrangement of the series $\sum_{n=1}^{\infty} a_n$, then

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n.$$ 

There are two special tests we have for absolute convergence, which turn out to be equivalent, meaning that one test will work if and only if the other test will work.

**Theorem 10.10** (The ratio test). Suppose $\sum_{n=1}^{\infty} a_n$ satisfies $a_n \neq 0$ for all $n \geq N$, for some integer $N$, and let $L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

2. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3. If $L = 1$, then no information is gained.

**Theorem 10.11** (The root test). Suppose $\sum_{n=1}^{\infty} a_n$ satisfies $a_n \neq 0$ for all $n \geq N$, for some integer $N$, and let $L = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} |a_n|^{1/n}$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If \( L > 1 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.

3. If \( L = 1 \), then no information is gained.

Both of these tests work by, essentially, telling us how far our series is from a geometric series.

**Example 99:** Does \( \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n} \) converge?

**Solution:** Our terms are all nonzero, so we may apply the ratio test. We have

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right| = \frac{1}{3} \left( \frac{n+1}{n} \right)^3,
\]

and taking the limit as \( n \to \infty \) gives us \( L = \frac{1}{3} \), using the notation of the ratio test. Because

\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3} < 1,
\]

it follows from the ratio test that the series converges absolutely.

**Example 100:** What about \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \)?

**Solution:** The presence of a factorial symbol is usually an indication that the ratio test may be applied. All our terms are again nonzero, so we may compute

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{(n+1)^{n+1}} \left( \frac{n}{n+1} \right)^n = \left( 1 - \frac{1}{n} \right)^n,
\]

which we recall from the definition of the exponential function converges to \( e^{-1} < 1 \). It now follows from the ratio test that the series is (absolutely) convergent.

**Example 101:** What about \( \sum_{n=1}^{\infty} \left( \frac{2n + 3}{3n + 4} \right)^n \)?
**Solution:** This time, the exponent $n$ indicates that we might want to try the root test (instead of the ratio test). Indeed,

$$|a_n|^{1/n} = \left| \left( \frac{2n + 3}{3n + 4} \right)^n \right|^{1/n} = \frac{2n + 3}{3n + 4} \to \frac{2}{3} < 1.$$ 

That is, $\lim_{n \to \infty} |a_n|^{1/n} = \frac{2}{3}$, and so by the root test, the series is absolutely convergent (and therefore convergent).

**Fact:** One has $\lim_{n \to \infty} n^{1/n} = 1$.

This fact will often be useful in various applications of the root test, and can be used without proof on any quiz or exam.

How can these tests fail? We consider three situations where no information is gained by applying the ratio test or the root test.

1. When $a_n = \frac{1}{n}$, we have

$$\lim_{n \to \infty} \left( \frac{1}{n+1} \right) = 1 \text{ (ratio test)}, \quad \lim_{n \to \infty} \left( \frac{1}{n} \right)^{1/n} = 1 \text{ (root test)},$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2. When $a_n = \frac{(-1)^n}{n}$, we have

$$\lim_{n \to \infty} \left| \left( \frac{(-1)^{n+2}}{n+1} \right) \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}}{n} \right|^{1/n} = 1,$$

and the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

3. When $a_n = \frac{1}{n^2}$, we have

$$\lim_{n \to \infty} \left( \frac{n+1}{n} \right)^2 = \lim_{n \to \infty} \left( \frac{1}{n^2} \right)^{1/n} = 1,$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges absolutely.
As we see, any type of convergence or divergence is possible when the limiting value $L$ in the ratio/root test is 1; in these situations, we have to use other tests to check convergence.

**Rough guide:** If an $n!$ appears in the term $a_n$, we might want to use the ratio test. If $n$ appears as an *exponent* in the term $a_n$, we might want to use the root test.

This closes our discussion on series of real numbers. Now, and perhaps for the last time in this course, we will take an existing construction and turn it on its head.

### 11 Power series

Recall the geometric series: if $x \in \mathbb{R}$ satisfies $-1 < x < 1$, then the series of *real numbers* $x^n, n = 0, 1, 2, \ldots$ is convergent. That is:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x| < 1.$$

Reversing the equality to write

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

changes our perspective: given the function $f(x) = \frac{1}{1-x}$, we have found a convergent series $\sum_{n=0}^{\infty} a_n$, whose terms depend on $x$ (so we can write $a_n = a_n(x)$), such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x);$$

we have found a way to represent the function $f(x)$ as a *series*, at least for certain values of $x$.

This motivates our next definition: given a series consisting not only of real numbers, but of nonnegative integer powers of some formal variable $x$, for what values of $x$ will the resulting series of *real numbers* converge?

**Remark:** A “formal variable” is one to which no *a priori* meaning has been assigned, so what we are saying here is this: given an expression

$$\sum_{n=0}^{\infty} a_n x^n,$$
where the $a_n$ are real numbers and $x$ is a “placeholder variable,” we obtain a series of real numbers by replacing the $x$ in the expression by some actual real number (so that e.g. plugging in $x = 1$ gives the series $\sum a_n$ of real numbers). We can then ask the question, for which real numbers $x$ does the series $\sum_{n=1}^{\infty} a_n x^n$ converge?

Definition: A power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \cdots,$$

where the $a_n$ are real numbers called the coefficients of the power series, and $c \in \mathbb{R}$ is called the centre of the series (the given power series is said to be centered on $c$).

Convention: All power series begin with $n = 0$, instead of the more usual $n = 1$ we say for real numbers; in case the expression starts from, say, $n = 2$, then the zeroeth and first coefficients of that power series are zero. Also, when we plug in $x = c$, the term corresponding to $n = 0$ is the formal expression $0^0$; in the context of power series, we will always assign this expression the value 1 (this is the same convention as with polynomials).

Picking up our previous discussion: if we define a function $f(x)$ by

$$f(x) := \sum_{n=0}^{\infty} a_n (x-c)^n,$$

then certainly $f(x)$ is well-defined for at least one point, namely, $x = c$; so the domain of $f(x)$ contains at least one real number. We might ask: what is the entire domain of $f(x)$?

Example 102: For what values of $x$ does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Solution: We remind ourselves that, because this expression starts at $n = 1$, then we consider this to be the power series

$$\sum_{n=0}^{\infty} a_n (x-3)^n, \quad \text{where } a_n = \begin{cases} 0 & \text{if } n = 0, \\ \frac{1}{n} & \text{if } n > 0. \end{cases}$$


We obviously have convergence when \( x = 3 \); otherwise, let us fix some real number \( x \), and apply the root test to the resulting series of real numbers. We have
\[
\left| \frac{(x - 3)^n}{n} \right|^{1/n} = \left| \frac{x - 3}{n^{1/n}} \right| \xrightarrow{n \to \infty} |x - 3|.
\]
If \( |x - 3| < 1 \), then the root test implies that the series converges absolutely; this happens exactly when
\[
-1 < x - 3 < 1 \iff 2 < x < 4.
\]
By the same token, the series will diverge if \( |x - 3| > 1 \), which happens if and only if
\[
x - 3 > 1 \iff x > 4 \ \text{OR} \ x - 3 < -1 \iff x < 2.
\]
We have now checked convergence at every real number except for \( x = 2 \) and \( x = 4 \). We call these the **boundary values**, and they **must be checked individually**. When \( x = 4 \) one has
\[
\sum_{n=1}^{\infty} \frac{(4 - 3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n},
\]
which diverges, and at \( x = 2 \) we have instead
\[
\sum_{n=1}^{\infty} \frac{(2 - 3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log 2.
\]
So our power series converges at \( x \) if and only if \( 2 \leq x < 4 \); that is, the domain of the function
\[
f(x) = \sum_{n=1}^{\infty} \frac{(x - 3)^n}{n}
\]
is \( x \in [2, 4) \).

**Example 103:** What about \( \sum_{n=0}^{\infty} n!x^n \)?

**Solution:** Again, we fix a real number \( x \). Because our terms are nonzero, we may apply the ratio test; the ratio of successive terms is
\[
\left| \frac{(n + 1)!x^{n+1}}{n!x^n} \right| = (n + 1)|x| \xrightarrow{n \to \infty} \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{otherwise}. \end{cases}
\]
From this, it follows that the series diverges everywhere **except** \( x = 0 \).
Example 104: Find the domain of the order zero Bessel function, defined

\[ J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n (n!)^2}. \]

Solution: Clearly if \( x = 0 \) then the series converges (to 1, which is the only nonzero term); we may now fix a real number \( x \) and assume without loss of generality that it is nonzero.

With the presence of the factorials, we may as well try the ratio test, which will now apply because our terms are nonzero. The quotient of successive terms is

\[ \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} ((n+1)!)^2} \right| = \frac{x^2}{2^2 (n+1)^2}, \]

which tends to zero as \( n \to \infty \), regardless of the value of \( x \) (this is because, in the limit, only \( n \) is changing, while \( x \) remains fixed). The ratio test then implies that \( J_0(x) \) converges for all nonzero real numbers, and we deduce that the domain of \( J_0(x) \) is all of \( \mathbb{R} \).

These three examples illustrate a fundamental property of power series, which is crucial to understand: a power series will converge on a single interval.

Theorem 11.1. Let \( \sum_{n=0}^{\infty} a_n (x-c)^n \) be a power series; then exactly one of the following statements is true.

1. The series converges at \( x = c \) and diverges everywhere else.

2. The series converges for all \( x \in \mathbb{R} \).

3. There exists a positive number \( R > 0 \) such that the series converges whenever \( |x-c| < R \), diverges whenever \( |x-c| > R \), and which may converge or diverge when \( x = c \pm R \).

Two comments about case 3.: one, we must investigate convergence at the endpoints \( c \pm R \) separately; there is no getting around this. Two, the value of \( R \) is called the radius of convergence; we define this quantity in case 1. to be \( R = 0 \), and in case 2. to be \( R = \infty \).

Examples 102, 103, and 104 are examples of cases 3., 1., and 2., respectively.
Let us draw a picture to better understand the situation in case 3:

\[ c - R \quad c \quad c + R \]

In the picture above, the blue line segment indicates the interval of convergence consisting of all those points \( x \) for which the series converges, and the red segments indicate the points at which the series diverges. The points \( c \pm R \) may or may not be in the interval of convergence: as such, there are four possibilities for this interval, namely

\[(c - R, c + R), \ (c - R, c + R], \ [c - R, c + R), \ or \ [c - R, c + R].\]

Technically, this picture is only true in case 3., but if we pretend for instance that \([c - 0, c + 0] = \{c\}, \ (c - \infty, c + \infty) = (\infty, \infty) = \mathbb{R},\)

then these formulas will also work in the other cases. Put another way: the interval of convergence is just the domain of the power series. Note that, by definition, the length of the interval of convergence is \(2R\), or twice the radius of convergence.

**Example 105:** We have already shown that the geometric series \(\sum_{n=0}^{\infty} x^n\) converges if and only if \(|x| < 1\); from this it follows immediately that the interval of convergence is \(x \in (-1, 1)\). In examples 102, 103, and 104, the respective intervals of convergence are \([2, 4], [0, 0] = \{0\},\) and \((\infty, \infty)\).

**Example 106:** Find the radius and interval of convergence of the power series

\[ f(x) = \sum_{n=0}^{\infty} \frac{n(x + 2)^n}{3^{n+1}}. \]

**Solution:** We observe that we can write

\[ f(x) = \frac{1}{3} \sum_{n=0}^{\infty} n \left( \frac{x + 2}{3} \right)^n, \]

which clearly converges when \(x = -2\) and otherwise has nonzero terms. Applying the root test, we have

\[ \left| n \left( \frac{x + 2}{3} \right)^n \right|^{1/n} = n^{1/n} \left( \frac{x + 2}{3} \right) \xrightarrow{n \to \infty} \frac{|x + 2|}{3}. \]
So the series will converge if
\[
\frac{|x + 2|}{3} < 1 \iff |x + 2| < 3 \iff -3 < x + 2 < 3 \iff -5 < x < 1,
\]
and it will diverge if \(|x + 2| > 3\), which happens when \(x > 1\) or \(x < -5\), and it remains only to check the points \(x = 1, -5\). In the former case we have
\[
f(1) = \sum_{n=0}^{\infty} \frac{n(1 + 2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n}{3} = \frac{1}{3} \sum_{n=1}^{\infty} n \to \infty = \infty,
\]
and in the latter case we have instead
\[
f(-5) = \frac{1}{3} \sum_{n=0}^{\infty} n \left(\frac{-5 + 2}{3}\right)^n = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n,
\]
which also diverges, by the divergence test. We deduce that \(f(x)\) has interval of convergence \((-5, 1)\), and therefore that its radius of convergence is \(R = \frac{1 - (-5)}{2} = 3\) (i.e., half the length of the interval).

**Fact:** If we write \(0 = \frac{1}{\infty}\) and \(\infty = \frac{1}{0}\), then whenever
\[
\lim_{n \to \infty} |a_n|^{1/n}
\]
exists, then the radius of convergence of the series \(\sum_{n=0}^{\infty} a_n(x - c)^n\) is
\[
R = \frac{1}{\lim_{n \to \infty} |a_n|^{1/n}}.
\]