Today, we begin by comparing two division problems.

**Division algorithm:** If \( a, b \) are any integers with \( b \neq 0 \), then there exists an integer \( q \) and an integer \( r \) with \( 0 \leq r < b \) such that

\[
a = bq + r; \text{ equivalently, } \frac{a}{b} = q + \frac{r}{b}.
\]

These numbers have the following names in this situation:

- The **dividend** is \( a \);
- The **divisor** is \( b \);
- The **quotient** is \( q \); and
- The **remainder** is \( r \).

In case \( a < b \), we will have the situation \( q = 0, r = a \).

**Example 65:** Find \( 1029 \div 79 \).

**Solution:**

\[
\begin{array}{c|c}
79 & 1029 \\
\hline
79 & \\
79 & 239 \\
237 & 2 \\
\end{array}
\]
We call 1029 the **dividend**, and 79 the **divisor**. We start reading the digits of the dividend from left to right, and with each successive digit we will try to apply the division algorithm to get a nonzero quotient.

In this case: the first digit of the dividend is 1, which is strictly smaller than 79. The division algorithm records a quotient of zero (with a remainder of 1), so we record zero and add another digit. We now perform the division algorithm to find $10 \div 79$, which again gives a zero quotient, and we add another digit.

This time, we have a nontrivial solution: that is, the division algorithm gives

$$102 = 79 \cdot 1 + 23.$$

We record the quotient 1, and subtract $79 \cdot 1$ from 102 to get our remainder 23.

Now, we add the next digit of our dividend to this remainder; the next digit is 9, so the figure 23 becomes 239. Our new dividend is 239, which satisfies

$$239 = 79 \cdot 3 + 2;$$

we record the quotient 3, subtract $237 = 79 \cdot 3$ from 239, and obtain the remainder 2. There are no more digits before the decimal place, so we have our answer; namely, $1029 = 79 \cdot 13 + 2$.

This approach generalizes to polynomials.

**Recall:** The degree of a polynomial, written $\deg f(x)$, is the largest integer that appears as an exponent in $f(x)$.

**Division algorithm for polynomials:** If $f(x), g(x)$ are any polynomials with $\deg g(x) \neq 0$, then there exists a polynomial $q(x)$ and an polynomial $r(x)$ with $\deg r(x) < \deg g(x)$ such that

$$f(x) = g(x)q(x) + r(x);$$

equivalently,

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}.$$

We retain the terminology from the integer case (so $f(x)$ is the divisor, $r(x)$ the remainder, etc.)

**Example 66:** Find $\frac{6x^3-2x+1}{2x+9}$.
Solution:

\[
\begin{array}{c|cc}
3x^2 & - \frac{27}{2}x & + \frac{239}{4} \\
\hline
2x + 9) & 6x^3 & - 2x & + 1 \\
- 6x^3 & - 27x^2 & \\
\hline & -27x^2 & - 2x \\
& 27x^2 & + \frac{243}{2}x \\
\hline & \frac{239}{2}x & + 1 \\
& - \frac{239}{2}x & - \frac{2151}{4} \\
\hline & - \frac{2147}{4}
\end{array}
\]

Our approach is largely the same, with a few slight tweaks: we start reading the terms of the dividend from highest power to least power, and with each successive term, try to apply the division algorithm for polynomials until we get a nonzero monomial (i.e. polynomial with exactly one term) quotient.

In our case, the highest power term is \(6x^3\), which already has nonzero quotient when divided by \(2x + 9\):

\[6x^3 = (2x + 9)(3x^2) - 27x^2.\]

We record a quotient \(3x^2\), and subtract \((2x + 9)(3x^2)\) from \(6x^3\) to get our remainder of \(-27x^2\). We add the next term of our dividend (i.e. \(-2x\)) to obtain the new dividend \(-27x^2 - 2x\), and apply the division algorithm again to get

\[-27x^2 - 2x = (2x + 9) \left( -\frac{27}{2}x \right) + \frac{239x}{2}.\]

We record the quotient \(-\frac{27}{2}x\), subtract \((2x + 9) \left( -\frac{27}{2}x \right)\) from \(-27x^2 - 2x\) to obtain \(\frac{239x}{2}\), and apply the algorithm once more:

\[\frac{239x}{2} = (2x + 9) \left( \frac{239}{4} \right) - \frac{2151}{4}.\]

We record the quotient \(\frac{239}{4}\), which is a constant, meaning the division process is done. We have shown

\[(6x^3 - 2x + 1) = (2x + 9) \left( 3x^2 - \frac{27}{2}x + \frac{239}{4} \right) - \frac{2151}{4},\]

and we are done.

Remark: Observe that we have left space in our long division symbol for the zero coefficients of our polynomial. That is: the coefficient of \(x^2\) in our dividend is zero.
and so we have left space above it in our long division symbol for the coefficient of $x^2$ in the quotient.

The point of introducing long division of polynomials is the following: the partial fraction decomposition of a rational function (i.e. a quotient $\frac{f(x)}{g(x)}$ of two polynomials) as I have defined it can only be found when the degree of $f$ is strictly smaller than the degree of $g$. When this is not the case (that is, when $\deg f(x) \geq \deg g(x)$), we use long division to write

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)},$$

where we will now have $\deg r(x) < \deg g(x)$. So, in our example above, we cannot find the partial fraction decomposition of

$$\frac{6x^3 - 2x + 1}{2x + 9},$$

but we can find the partial fraction decomposition of the equivalent expression

$$(3x^2 - \frac{27}{2}x + \frac{239}{4}) + \frac{-2151}{4(2x + 9)},$$

because it is in the correct form.

**Example 67:** Find $\frac{4x^4 + 6x^2 + 28x - 3}{x^2 + x + 1}$.

**Solution:**

We perform the same algorithm as in the previous example. Partial fraction decomposition (or PFD) is a tool we use to clean up rational functions to make them easier to integrate; we saw one example at the end of the last lecture. In general, the process will be this:

$$\begin{align*}
4x^2 - 4x + 6 & \quad \Rightarrow \quad 4x^4 - 4x^3 - 4x^2 \\
& \quad \Rightarrow \quad -4x^3 + 2x^2 + 28x \\
& \quad \Rightarrow \quad 4x^3 + 4x^2 + 4x \\
& \quad \Rightarrow \quad 6x^2 + 32x - 3 \\
& \quad \Rightarrow \quad -6x^2 - 6x - 6 \\
& \quad \Rightarrow \quad 26x - 9
\end{align*}$$
1. Start with a rational function $\frac{f(x)}{g(x)}$, where $f, g$ are polynomials and $\deg g(x) > 0$.

1’. If $\deg f(x) \geq \deg g(x)$, perform long division to obtain $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$, and return to step 1 with $\frac{r(x)}{g(x)}$ instead of $\frac{f(x)}{g(x)}$.

2. Write the PFD of our rational function according to the rules laid out below.

3. Integrate the resulting PFD according to the formulas

$$\int \frac{dx}{Ax + B} = \frac{1}{A} \log |Ax + B| + C,$$

$$\int \frac{dx}{(Ax + B)^k} = \frac{(Ax + B)^{-k+1}}{A(-k+1)} \quad \text{for} \quad k > 1,$$

and using substitutions for higher degrees.

The rules for computing the PFD of $\frac{f(x)}{g(x)}$, with $\deg f < \deg g$, are as follows.

**Warning!/Attention!/注意!/¡Atención!/Achtung!/주의!**

Any deviation from the following rules will lead to you being unable to solve for the unknown constants in the partial fraction decomposition. The existence of a partial fraction decomposition of the sort we have described is dependent on the following rules being followed exactly.

**Rule One:** If we can write $g(x)$ as the product of distinct linear terms, so

$$g(x) = (a_1 x + b_1) \cdots (a_k x + b_k),$$

then the PFD is

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k},$$

where each $A_i$ are constants.

**Example 68:** Because $x^3 - x = x(x + 1)(x - 1)$, we have

$$\frac{1}{x^3 - x} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 1} = \frac{A(x+1)(x-1) + Bx(x-1) + Cx(x+1)}{x(x+1)(x-1)} = \frac{(A + B + C)x^2 + (-B + C)x - A}{x^3 - x}.$$
A + B + C = 0 (coefficient of $x^2$), $-B + C = 0$ (coefficient of $x^1 = x$), and $-A = 1$ (coefficient of $x^0 = 1$). Solving this system gives

\[ A = -1, \quad B = C = \frac{1}{2}, \]

and we have

\[ \frac{1}{x^3 - x} = \frac{-1}{x} + \frac{1/2}{x + 1} + \frac{1/2}{x - 1}. \]

This is our partial fraction decomposition.

**Rule Two:** If \((ax + b)^k\) divides the denominator, where \(k > 1\), then the PFD contains the terms

\[ \frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}, \]

where the \(A_i\) are constants.

**Example 69:** Because \(x^4 - 3x^3 + 3x^2 - x = x(x - 1)^3\), we have

\[
\frac{1}{x^4 - 3x^2 + 3x - x} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3} \\
= \frac{A(x - 1)^3 + Bx(x - 1)^2 + Cx(x - 1) + Dx}{x(x - 1)^3} \\
= \frac{(A + B)x^3 + (-3A - 2B + C)x^2 + (3A + B - C + D)x - A}{x^4 - 3x^2 + 3x^2 - x}. \\
\]

We now aim to solve the system of equations

\[
1. A + B = 0; \quad 2. -3A - 2B + C = 0; \quad 3. 3A + B - C + D = 0; \quad 4. -A = 1. \\
\]

From the fourth equation we deduce \(A = -1\), so from the first we have \(B = 1\); plugging these into the second equation gives \(C = -1\), and finally from equation 3. we get \(D = 1\). Thus

\[
\frac{1}{x^4 - 3x^2 + 3x - x} = \frac{-1}{x} + \frac{1}{x - 1} - \frac{1}{(x - 1)^2} + \frac{1}{(x - 1)^3}, \]

and we can now integrate the function with ease.

**Rule Three:** If \(ax^2 + bx + c\) is an irreducible quadratic (meaning \(b^2 - 4ac < 0\)) which divides \(g(x)\), and \((ax^2 + bx + c)^2\) does not divide \(g(x)\), then the partial fraction decomposition of \(\frac{f(x)}{g(x)}\) contains the term

\[ \frac{Ax + B}{ax^2 + bx + c}, \]

where \(A\) and \(B\) are constants.
Example 70: Because $x^4 + x^2 = x^2(x^2 + 1)$, and $0^2 - 4(1)(1) < 0$, we know that $x^2 + 1$ is an irreducible quadratic; so,

$$\frac{1}{x^4 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} = \frac{Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2}{x^2(x^2 + 1)} = \frac{(A + C)x^3 + (B + D)x^2 + (A)x + B}{x^4 + x^2},$$

and we must solve the system (exercise)


It turns out this system has solution $A = C = 0, B = -D = 1$, and hence

$$\frac{1}{x^4 + x^2} = \frac{1}{x^2} - \frac{1}{x^2 + 1},$$

and we are done.

Rule Four, and the last: If $ax^2 + bx + c$ is an irreducible quadratic and $(ax^2 + bx + c)^k$ divides $g(x)$ for $k > 1$, then the PFD of $\frac{f(x)}{g(x)}$ contains the terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k},$$

where the $A_i, B_i$ are constants.

Example 71: We have

$$x^6 + x^5 + 8x^4 + 8x^3 + 16x^2 + 16x = (x^2 + x)(x^4 + 8x^2 + 16x) = (x + 1)(x^2 + 4)^2.$$

Because $x^2 + 4$ is irreducible (check this!), we have

$$\frac{x^2 + x + 1}{x^6 + x^5 + 8x^4 + 8x^3 + 16x^2 + 16x} = \frac{A}{x} + \frac{B}{x + 1} + \frac{Cx + D}{x^2 + 4} + \frac{Ex + F}{(x^2 + 4)^2} = \frac{1}{x(x + 1)(x^2 + 4)^2}\left((A + B + C)x^5 + (A + C + D)x^4 + (8A + 8B + 4C + D + E)x^3 + (8A + 4C + 4D + E + F)x^2 + (16A + 16B + 4D + F)x + 16A\right).$$

This time, we have the system of equations (numbered by the corresponding exponent of $x$):
0. $16A = 1$;
1. $16A + 16B + 4D + F = 1$;
2. $8A + 4C + 4D + E + F = 1$;
3. $8A + 8B + 4C + D + E = 0$;
4. $A + C + D = 0$;
5. $A + B + C = 0$.

Solving this system (exercise) gives

$$A = \frac{1}{16}, \quad B = D = \frac{-1}{25}, \quad C = \frac{-9}{400}, \quad E = \frac{-1}{20}, \quad F = \frac{4}{5}.$$ 

Thus, the PFD is

$$\frac{x^2 + x + 1}{x^6 + x^5 + 8x^4 + 8x^3 + 16x^2 + 16x} = \frac{1/16}{x} + \frac{(-1/25)}{x + 1} + \frac{(-9/400)x - (1/25)}{x^2 + 4} + \frac{(-1/20)x + (4/5)}{(x^2 + 4)^2},$$

and we are done.

We reiterate: every partial fraction decomposition are combinations of the above four rules which must be followed exactly, combined with the protocol:

1. Start with a rational function $\frac{f(x)}{g(x)}$.
2. If $\deg f(x) \geq \deg g(x)$, perform long division to write

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)},$$

where now $\deg r(x) < \deg g(x)$.

3. Apply The Rules to find the PFD of $\frac{f(x)}{g(x)}$, or $\frac{r(x)}{g(x)}$ if step 2. was necessary.

Finally, one more reminder: partial fraction decompositions have nothing to do with integration, and is simply an algebraic trick to make the integrals we take later more simple. We frequently exploit PFDs in other situations.

Onto our next topic: improper integration.
8 Improper integrals and approximation

Suppose \( f(x) \) is a function which is continuous for all nonnegative real numbers. What would be a reasonable way to define

\[
\int_0^\infty f(x) \, dx?
\]

We defined the definite integral only over intervals of finite length (i.e. intervals \([a, b]\), where both \(a\) and \(b\) are real numbers), and so we can’t use the same definition to assign meaning to this new expression. It turns out that the way that fits most analogously with our other definitions is this one: we define

\[
\int_a^\infty f(x) \, dx := \lim_{A \to \infty} \int_a^A f(x) \, dx,
\]

provided that the limit exists. Similarly, we will define

\[
\int_{-\infty}^b f(x) \, dx := \lim_{A \to -\infty} \int_A^b f(x) \, dx,
\]

provided the limit exists. Finally, if both of these limits exist, we will define

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^c f(x) \, dx + \int_c^{\infty} f(x) \, dx,
\]

for any \(c \in \mathbb{R}\).

**Definition:** If the limit \( \lim_{A \to \infty} \int_a^A f(x) \, dx \) exists, then we say that the integral \( \int_a^\infty f(x) \, dx \) is **convergent**; otherwise it is **divergent** (and similarly for integrals on \((-\infty, b)\) or \((-\infty, \infty)\)). In case the improper integral is divergent, then the expression \( \int_0^\infty f(x) \, dx \) has no meaning.

**Example 72:** Determine whether or not \( \int_1^\infty \frac{dx}{x^2} \) is convergent.

**Solution:** There is only one distinction between an improper integral and a definite integral, and it is a slight one: we will start by replacing our integral by

\[
\lim_{A \to \infty} \int_1^A \frac{dx}{x^2}.
\]
and that’s it! As $A$ tends to infinity, each integral on the right-hand side is an ordinary definite integral, and we can use our usual tricks with it. We know

$$
\int_1^A \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^A = 1 - \frac{1}{A},
$$

and so by definition

$$
\lim_{A \to \infty} \int_1^A \frac{dx}{x^2} = \lim_{A \to \infty} \left(1 - \frac{1}{A}\right) = 1.
$$

That is: we have shown that

$$
\int_1^\infty \frac{dx}{x^2} = 1,
$$

and so in particular this improper integral is convergent.

\[\square\]

**Example 73:** For what values of $p \in \mathbb{R}$ is the improper integral $\int_1^\infty \frac{dx}{x^p}$ convergent?

**Solution:** We start (as always) by writing

$$
\int_1^\infty \frac{dx}{x^p} = \lim_{A \to \infty} \int_1^A \frac{dx}{x^p},
$$

and then split into cases.

**Case one:** If $p = 1$, then $\frac{1}{x^p} = \frac{1}{x}$, and so

$$
\lim_{A \to \infty} \int_1^A \frac{dx}{x^p} = \lim_{A \to \infty} \log |x|_1^A = \lim_{A \to \infty} \log A = \infty,
$$

and so the integral diverges in this case.

**Case two:** In case $p \neq 1$, then the antiderivative of $x^{-p}$ is $\frac{x^{1-p}}{-p+1}$, and so

$$
\lim_{A \to \infty} \int_1^A \frac{dx}{x^p} = \lim_{A \to \infty} \left[\frac{x^{1-p}}{-p+1}\right]_1^A = \lim_{A \to \infty} \left(\frac{A^{1-p}}{-p+1} - \frac{1}{-p+1}\right) = -\frac{1}{p-1} \lim_{A \to \infty} (A^{p+1} - 1).
$$

We recall from example 9 that

$$
\lim_{n \to \infty} n^r = \begin{cases} 
0 & \text{if } r < 0, \\
\infty & \text{if } r > 0.
\end{cases}
$$
So, if \( p > 1 \), then \(-p + 1 < 0\) and so the interval converges, to \( \frac{1}{p-1} \); if \( p < 1 \), then the integral diverges. Thus, we have shown that

\[
\int_1^\infty \frac{dx}{x^p}
\]

covers if and only if \( p > 1 \) ♣

The result we have from this example is known as the \textbf{(integral) } \( p \)-test.

There is another sort of improper integral we have to deal with: definite integrals in which the integrand is \textit{not} continuous at one endpoint, or both endpoints. We deal with this issue in the same way, namely, by taking limits.

\textbf{Definition:} Suppose \( f(x) \) is continuous for all \( a < x \leq b \), and is discontinuous at \( x = a \). Then we define

\[
\int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx,
\]

where we read the limit on the right-hand side as “the limit as \( t \) tends to \( a \) from above,” \textit{provided that this limit exists}.

\textbf{Example 74}: Find \( \int_0^1 \log x \, dx \).

\textbf{Solution}: We know that \( \log x \) is discontinuous at \( x = 0 \), so by our definition

\[
\int_0^1 \log x \, dx = \lim_{t \to 0^+} \int_t^1 \log x \, dx.
\]

From example 51 we know that

\[
\int \log x \, dx = x \log x - x + C,
\]

and so

\[
\int_0^1 \log x \, dx = \lim_{t \to 0^+} [x \log x - x]_{x=t}^1 = \lim_{t \to 0^+} (t - t \log t) - 1.
\]

For this limit, we apply L’Hôpital’s rule:

\[
\lim_{t \to 0^+} \left( t - \frac{\log t}{1/t} - 1 \right) = \lim_{t \to 0^+} \left( t - \frac{1/t}{-1/t^2} - 1 \right) = -1.
\]
From this we deduce
\[ \int_0^1 \log x \, dx = -1, \]
and we are done.

In general we have the following definitions:

- If \( f(x) \) is continuous on \([a, b)\), and not at \( b \), then define
  \[ \int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx, \]
  provided the limit exists.

- If \( f(x) \) is continuous on \((a, b]\), and not at \( a \), then define
  \[ \int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx, \]
  provided the limit exists.

- If \( f(x) \) is discontinuous at \( c \in (a, b) \), and is continuous on both \([a, c)\) and \((c, b]\), then define
  \[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx, \]
  provided both integrals on the right converge.

**Example 75:** Determine whether or not \( \int_0^3 \frac{dx}{x - 1} \) converges.

**Solution:** The integrand is well-defined and continuous everywhere apart from \( x = 1 \). According to our definition, therefore, we write
\[ \int_0^3 \frac{dx}{x - 1} = \int_0^1 \frac{dx}{x - 1} + \int_1^3 \frac{dx}{x - 1}, \]
and consider the two integrals on the right. We note that
\[ \int_0^1 \frac{dx}{x - 1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x - 1} = \lim_{t \to 1^-} [\log |x - 1|]_x^1 = \lim_{t \to 1^-} \log(1 - t), \]
which diverges to $-\infty$. Because the integral $\int_{0}^{1} \frac{dx}{x - 1}$ does not converge, we see that $\int_{0}^{3} \frac{dx}{x - 1}$ cannot converge either.

Note that, in the last example, we didn’t even have to look at the integral $\int_{1}^{3} \frac{dx}{x - 1}$.

We have two cheap ways to check the convergence of an improper integral; namely, the comparison test.

**Theorem 8.1** (The comparison test). Suppose $f(x), g(x)$ are continuous on $[a, \infty)$, and that $f(x) \geq g(x) \geq 0$.

1. If $\int_{a}^{\infty} f(x) \, dx$ converges, then $\int_{a}^{\infty} g(x) \, dx$ converges.
2. If $\int_{a}^{\infty} g(x) \, dx$ diverges, then $\int_{a}^{\infty} f(x) \, dx$ diverges.

**Caution:** We can *never* deduce the value of a convergent improper integral using the comparison test, apart from knowing it is finite.

**Example 76:** We know that $e^x > 0$ for all $x$, and so in particular

$$\frac{1}{x} < \frac{1}{x} + \frac{e^{-x}}{x} \text{ for all } x \geq 1.$$ 

Hence, because $\int_{1}^{\infty} \frac{dx}{x}$ diverges by the $p$-test, we see by the comparison test that $\int_{1}^{\infty} \frac{1 + e^{-x}}{x}$ diverges also.

Next time we will move onto approximations and differential equations.