See the syllabus at http://www.math.ubc.ca/~belked/lecturenotes/101.3/m101syllabus.pdf for information on evaluation, office hours, expectations, etc.

1 Sequences

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers, and let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ be the set of nonnegative integers. By a sequence of real numbers, we mean a list of real numbers indexed by $\mathbb{N}$ (or $\mathbb{N}_0$). We can write out a sequence like

$$(a_1, a_2, a_3, \ldots) \text{ or } (a_n)_{n \in \mathbb{N}} \text{ or } \{a_n\}_{n=1}^{\infty} \text{ or } (a_n).$$

We do not want to confuse the sequence $(a_n)_{n \in \mathbb{N}}$ with the set $\{a_n : n \in \mathbb{N}\}$. A sequence differs from a set in the following ways:

1. a set is unordered, while a sequence is ordered.
2. a set has no repetition, while a sequence may have repetition.
3. given a sequence

$$(a_1, a_2, a_3, \ldots),$$

we can always obtain the set of points $\{a_n : n \in \mathbb{N}\}$, but given a set, we cannot generally form a sequence in any meaningful way.

As such, it is important to be careful with sequence notation (when you write $\{a_n\}$, what are you talking about?)

**Examples of sequences:**
1. The sequence \((1, 2, 3, 4, 5, \ldots)\), whose \(n\)th term is \(n\).
2. The constant sequence \((1, 1, 1, 1, \ldots)\).
3. The “arithmetic” sequence \((1, 4, 7, 10, 13, \ldots)\).
4. The “geometric” sequence \((3, 12, 48, 196, 784, \ldots)\).
5. The Fibonacci sequence \((0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots)\).
6. The “alternating” sequence \((-1, 1, -1, 1, -1, 1, \ldots)\).
7* The sequence \((2, 3, 5, 7, 11, 13, 17, \ldots)\) of prime numbers.
8* The Kolakoski sequence \((1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 1, 1, \ldots)\).

There are many ways to describe a sequence. One way is to list its terms, as we have just done. Another way is to define it recursively, by stating the first (few) term(s), and giving the rest in terms of those which come before. For the same examples, we have recursive definitions:

1. \(a_1 = 1; a_n = a_{n-1} + 1\) for \(n \geq 2\).
2. \(a_1 = 1; a_n = a_{n-1}\) for \(n \geq 2\).
3. \(a_1 = 1; a_n = a_{n-1} + 3\) for \(n \geq 2\).
4. \(a_1 = 3; a_n = 4a_{n-1}\) for \(n \geq 2\).
5. \(a_1 = 0, a_2 = 1; a_n = a_{n-1} + a_{n-2}\) for \(n \geq 3\).
6. \(a_1 = -1; a_n = -a_{n-1}\) for \(n \geq 2\).

Sometimes, it is possible to give a closed form expression for \((a_n)\); that is, we can write down, and compute, a function \(f : \mathbb{N} \to \mathbb{R}\) such that \(f(n) = a_n\). For the same examples, we give closed form expressions:

1. \(a_n = n\);
2. \(a_n = 1\);
3. \(a_n = 3n - 2\);
4. \(a_n = 4^{n-1} \cdot 3\);
5. \(a_n = \frac{\phi^n - \psi^n}{\sqrt{5}}\), where \(\phi = \frac{1 + \sqrt{5}}{2}\) and \(\psi = \frac{1 - \sqrt{5}}{2}\).

6. \(a_n = (-1)^n\).

**Exercise 1:** try to deduce these closed form definitions from the recursive definitions.

**Exercise 2:** compute the first 4 terms of the sequence \(a_n = (-1)^n 2^n / n!\).

**Remark:** Every sequence \((a_n)\) can, in fact, be considered a function \(a(n)\) whose domain is \(\mathbb{N}\) and which satisfies \(a(n) = a_n\); it is not, however, always easy to write down the formula.

There are two ways we commonly visualize the points in a sequence. One way (left) is to graph the sequence \(a_n\) as the points \((n, a_n)\) in the Cartesian plane; because our sequence has infinitely many terms, this graph will stretch out to the right, forever. This sort of picture emphasizes the connection between the sequence \((a_n)\), and the function \(a(n) = a_n\): the points in my picture are exactly the integer points of the graph of \(f(x) = 1/x\).

Another way (below) is to plot only the values \(a_n\) at their correct positions on the number line. This time, some sequences will be able to “fit” into a finite-length subset of the number line, while others (such as \(a_n = n\)) will still need an infinitely long interval.

This sort of picture helps us understand if the sequence is bounded, or has a limit, or other properties we will learn about soon.

The usual arithmetic operations of addition, subtraction, multiplication, and division (by NONZERO real numbers) extend naturally to sequences by “component-wise” operations. For instance, if \((a_n), (b_n)\) are two sequences, then:

- \((a_n) \pm (b_n) = (a_n \pm b_n) = (a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots)\);
\[ (a_n)(b_n) = (a_nb_n) = (a_1b_1, a_2b_2, a_3b_3, \ldots); \]

and, if we assume that \((b_n)\) consists only of NONZERO terms, then

\[ (a_n)/(b_n) = (a_n/b_n) = (a_1/b_1, a_2/b_2, a_3/b_3, \ldots) \]

**Example 1:** Suppose \(a_n = n + 2\) and \(b_n = \frac{1}{5^n}\); then

\[
(a_nb_n) = \left((n + 2) \cdot \frac{1}{5^n}\right) = \left(\frac{n + 2}{5^n}\right) = \left(\frac{3}{5}, \frac{4}{25}, \frac{5}{125}, \frac{6}{625}, \ldots\right)
\]

**Example 2:** Suppose \((c_n)\) is the sequence of daily costs of a factory in dollars (so the factory’s expense in dollars on day \(n\) is \(c_n\)); similarly, suppose \(r_n\) is the factory’s daily revenue in dollars. Then the sequence of daily profits is given by

\[
(p_n) = (r_n) - (c_n) = (r_n - c_n).
\]

When talking about a sequence, it is helpful to identify some properties it may have. For example, suppose a sequence \((a_n)\) satisfies the inequalities

\[ a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots; \]

such a sequence is said to be **increasing**. If instead we replace all the inequalities with **strict** inequalities, so

\[ a_1 < a_2 < a_3 < a_4 < \cdots, \]

then \((a_n)\) is said to be **strictly increasing**. Similarly, a sequence satisfying the inequalities

\[ a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \]

is said to be **decreasing**, and one satisfying

\[ a_1 > a_2 > a_3 > a_4 > \cdots \]

is said to be **strictly decreasing**. When we say a sequence is **monotone**, we mean that it satisfies one of these four properties (so its terms all “go in the same direction”).
Question: when is a sequence both increasing and decreasing?

Another natural question we might ask about our sequence is: do the terms of my sequence stay small, or can they become arbitrarily large?

Aside: to say that something (a function \( f \), say) can become arbitrarily large means that, if you give me any real number \( M \) at all, then I can find a value \( x \) satisfying \( f(x) > M \).

Definition: A sequence \((a_n)\) is said to be bounded above if there exists a number \( M \in \mathbb{R} \) such that 
\[ a_n \leq M \text{ for all } n \in \mathbb{N}. \]

Similarly, \((a_n)\) is bounded below if there exists some \( m \in \mathbb{R} \) satisfying 
\[ m \leq a_n \text{ for all } n \in \mathbb{N}. \]

If \((a_n)\) is both bounded above and bounded below (not just one or the other), it is said to be bounded.

Roughly speaking: a sequence is bounded if, when we plot it on the number line as above, all of its points lie in some interval of finite length; or, if we graph it on the Cartesian plane like a function, the \( y \)-values of its graph lie in some finite-length interval.

Fact: A sequence \((a_n)\) is bounded if and only if there exists some \( K > 0 \) such that 
\[ |a_n| < K \text{ for all } n \in \mathbb{N}; \text{ that is, } -K < a_n < K \text{ for all } n \in \mathbb{N}. \]

Proof. Exercise. \( \square \)

Using the same examples as before:

1. is bounded below.
2. is bounded.
3. is bounded below.
4. is bounded below.
5. is bounded below.
6. is bounded.
An example of a “totally unbounded” sequence (not a real term) is

\[(a_n) = ((-1)^n \cdot n) = (-1, 2, -3, 4, -5, 6, \ldots)\]

Very often, we are interested not only in the sequence itself, but also in any values it may be “approaching.”

**Example 3:** Suppose a ball is manufactured which, on every bounce, attains exactly 60% the height of the bounce it previously attained (see graph at right). If \(a_n\) is the ball’s greatest height, in metres, between the \((n - 1)\)th bounce and the \(n\)th bounce, then we have that

\[(a_n) = (1, 0.6, 0.6^2, 0.6^3, \ldots).\]

Clearly, \((a_n)\) is a (strictly) decreasing sequence of positive real numbers.

Real-world experience (always dangerous in mathematics) tells us that the ball should stop bouncing “eventually.” How can we make this notion more rigorous?

First of all: we notice that, given any positive height (of \(h\) metres, say), then only finitely many of our bounces can possibly be higher than \(h\). A simple calculation confirms this:

\[(0.6)^n \geq h \iff n \log(0.6) \geq \log h \iff n \leq \frac{\log h}{\log(0.6)}\]

Note that the inequality in the last statement flips, because we have divided by \(\log(0.6)\), which is negative). That is, given any arbitrary distance \(h > 0\), only finitely many terms of our sequence are at least \(h\) away from zero.

This example motivates our definition of a *limit*, to be seen in the next lecture.

**More examples:**

1. Let \(s_n\) be the greatest number of cells that can be obtained by mutually joining each of \(n\) distinct points on the circumference of a circle:

\[
\begin{align*}
n = 1, s_1 &= 1 \\
n = 2, s_2 &= 2 \\
n = 3, s_3 &= 4 \\
n = 4, s_4 &= 8
\end{align*}
\]
We are tempted to say \( s_n = 2^{n-1} \), however,

\[
\begin{align*}
  n = 5, & \quad s_5 = 16 \\
  n = 6, & \quad s_5 = 31 \\
  n = 7, & \quad s_5 = 57
\end{align*}
\]

It is important to verify any guess of a formula/closed form expression for a sequence; as this example shows, patterns do not always continue in the most obvious way. It turns out in this case that the closed form is

\[
s_n = \frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.
\]

**Question:** Is this sequence bounded? Monotone? *(the closed form doesn’t really help us here).*

2. Let \( e_n = \left(1 + \frac{1}{n}\right)^n \); we compute the first few terms:

\[
e_1 = 2, \quad e_2 = \left(\frac{3}{2}\right)^2, = 2.24, \quad e_3 = \left(\frac{4}{3}\right)^3 = 2.370.
\]

By the binomial theorem,

\[
e_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!n^k}.
\]

Just by looking, it is difficult to say much about this sequence (whether or not it is monotone, bounded, etc.) Even investigating the function \((1 + x^{-1})^x\) does not help:

\[
y = \left(1 + \frac{1}{x}\right)^x, \text{ so } \log y = x \left(1 + \frac{1}{x}\right),
\]

and so by implicit differentiation we have

\[
\frac{1}{y} \cdot \frac{dy}{dx} = \log \left(1 + \frac{1}{x}\right) + x \left(1 + x^{-1}\right) \cdot \left(-\frac{1}{x^2}\right) = \log \left(1 + \frac{1}{x}\right) - \frac{1}{1 + x}.
\]

We will return to this sequence in due course.
3. Consider the sequence

\[ x_n = \begin{cases} 
  n! & \text{if } n \text{ is odd,} \\
  \frac{1}{n} & \text{if } n \text{ is even.}
\end{cases} \]

This sequence neither approaches one value, nor blows up to infinity. However, it is clear that the subsequence of odd terms

\[ y_n = x_{2n-1} = (1!, 3!, 5!, 7!, 9!, \ldots) \]

goes off to infinity, and that the subsequence of even terms

\[ z_n = x_{2n} = \left( \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots \right) \]

diminishes to zero.

4. The decimal expansions of real numbers naturally give rise to several sequences. For instance, let \((d_n)\) be the sequence whose \(n\)th term is the truncation of \(\sqrt{2}\) at the \(n\)th decimal place; so

\[ d_1 = 1.4, \quad d_2 = 1.41, \quad d_3 = 1.414, \quad d_4 = 1.4142, \]

and so on. Notice that:

(a) each \(d_n\) is a rational number, but \(\sqrt{2}\) is not a rational number.

(b) the sequence is increasing (though not necessarily strictly so!) and it is bounded (by what? below? above?)

(c) if \(p_n\) denotes the \(n\)th digit to the right of the decimal place in the decimal expansion of \(\sqrt{2}\), then we can write

\[ d_n = 1 + \sum_{k=1}^{n} p_k \cdot 10^{-k}. \]

5. In the vein of the last example: given any sequence \((r_n)\) of numbers in the set \(\{0, 1, 2, \ldots, 9\}\), we can construct a new sequence

\[ R_n = \sum_{k=1}^{n} r_k \cdot 10^{-k}. \]

For instance, by taking \(r_n = 9\) for every \(n\), we have

\[ R_1 = 0.9, \quad R_2 = 0.99, \quad R_3 = 0.999, \quad \ldots, \quad R_N = 0.99\ldots9, \quad \ldots \]