These notes are intended to provide some background information on how we will be learning calculus this semester.

1 Terminology and notation

One chief concern mathematicians have is to be able to communicate clearly and precisely their ideas; so, it is to our advantage to know from the beginning how to start communicating like mathematicians. Let’s start with some notation:

- I will assume everyone is familiar with the symbols $+,-,\pm,=,<,\leq,\times$, and $\sqrt{}$.
- log and ln are interchangeable in my class, and both mean the natural logarithm.
- $\mathbb{R}$ – the set of all real numbers.
- $\mathbb{N}$ – the set of all natural numbers (positive whole numbers).
- $\mathbb{Z}$ – the set of all integers (positive and negative whole numbers, and zero).
- $\mathbb{Q}$ – the set of all rational numbers (ratios $a/b$ of integers with $b \neq 0$).
- $\epsilon$ – a symbol meaning “is a member of” regarding elements of sets; its opposite is $\notin$ (so $\pi \in \mathbb{R}$, but $\pi \notin \mathbb{Z}$).
- $\sum$ – the summation symbol, indicating that we are going to add whatever is written to its right.
• ! – the factorial symbol; if \( n \geq 1 \) is a whole number, then \( n! \) is the product \( 1 \times 2 \times 3 \times \cdots \times n \), and \( 0! = 1 \) by convention.

Finally, we briefly recall how to define sets by “predicates” (or properties): I read the expression \( A = \{ x \in \mathbb{R} : \sin x = \frac{1}{3} \} \) as “\( A \) is the set of all real numbers \( x \) such that the sine of \( x \) equals one-third,” and so \( x \in A \) if and only if \( \sin x = \frac{1}{3} \).

Now, here are some English words that have precise mathematical meanings:

• **expression** – a bunch of mathematical symbols, considered as a whole.

• **equation** – two expressions joined by an = sign.

• **function** – a rule for assigning, to each element of one set \( X \), exactly one element of another set \( Y \). If our function is called \( f \), then \( X \) is called the **domain** of \( f \), and \( Y \) is called the **codomain**; the subset \( f(X) \) of \( Y \) consisting of all values that \( f \) takes is called the **range**. We write \( f : X \to Y \) as shorthand for “\( f \) is a function with domain \( X \) and codomain \( Y \).”

• **variable** – a symbol representing a number, which can be arbitrary, unspecified, or unknown.

• **constant** – a symbol representing a number, which has a fixed (though possibly unknown) value.

• **summand** – a quantity which is summed.

• **integrand** – a function which is being integrated.

• **if and only if** – used to join two things that are either both true, or both false.

• **a priori** – without looking too deeply, at first glance, etc. (from the Latin “from the earlier”)

Some examples of how these words are used:

1. We remember the **quadratic formula** from high school: if \( ax^2 + bx + c = 0 \) is a quadratic equation, then

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]
In the equation $ax^2 + bx + c = 0$, the letters $a, b, c$ are constants with $a \neq 0$ (so the equation is actually quadratic), and $x$ is the variable, here representing the number(s) which make the equation true. The quadratic formula gives us an expression for this value of $x$, namely, $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

2. The function $f(x) = 2x^2 + 3x - 1$ is a rule sending the real number $x$ to the real number $2x^2 + 3x - 1$; so the domain of $f(x)$ is $\mathbb{R}$, and a priori the codomain is also $\mathbb{R}$. With a bit more work, we can show that the image of $f(x)$ is the set of all real numbers greater than or equal to $-\frac{17}{8}$.

3. Let $x$ be a real number, and suppose I want to add the powers of $x$, say from the zeroth power to the 11th power; I could write

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11},$$

or I could save space by writing

$$1 + x + x^2 + \cdots + x^{11}$$

and hoping that the reader understands; or I could write

$$\sum_{i=0}^{11} x^i;$$

and read it as “the sum from $i$ equals zero to eleven of $x$ to the power $i$.”

The bold part of this sentence tells me the indexing set (namely, \{0, 1, 2, \ldots, 11\}); the slanted part of this sentence tells me what the $i$th summand is, as $i$ ranges over all values of the indexing set. The symbol $i$ here is a dummy variable (or placeholder variable), meaning its purpose is to “hold the place” of whatever indexing value goes into the expression. This means

$$\sum_{i=0}^{11} x^i = \sum_{*=0}^{11} x^* = \sum_{\otimes=0}^{11} x^\otimes;$$

observe that the variable $x$ can not change without affecting the meaning of the expression.

We will see examples of integrands later in the course.

Finally, let’s look at some expressions and see how a mathematician would say them out loud.
• $\int_1^5 \sin x \, dx$ – “the integral from 1 to 5 of sine eks dee-eks”

• $\sum_{j=9}^{22} \frac{x^j}{j!}$ – “the sum from jay equals nine to twenty-two of eks to the jay, all over jay factorial”

• $\int e^{-t^2} \, dt$ – “the integral (or antiderivative) of ee to the minus tee squared dee tee”

2 Different kinds of obstacles

In order to maximize our ability to learn mathematics in a constrained period of time, it is often extremely helpful to be able to identify our own strengths and weaknesses, in order to focus our study time most effectively. In my own experience, I have found that most the incorrect answers I have given on exams in my life fall into one of the following categories:

1. Lack of comprehension: I haven’t studied the topic, or I need someone to explain it to me. This is the most important category to address.

2. Lack of time: I haven’t practiced the topic enough, or am not familiar enough with it, to solve the problem quickly, but I could solve it with some more time. Time is usually lost here straining my memory for the correct method, or working out first a related example I do remember, to see if that helps.

3. Silly mistake: I came up with a solution that would be correct, except for a momentary mental lapse. For instance, given the two longest side lengths of an isosceles (not right) triangle, I might blink and use the Pythagorean theorem $x^2 + y^2 = z^2$ to find the length of the remaining side.

4. Figure goblins: These are the small beings which dance across the page as I write the exam, making me misread the + on the line above as a −, and making me obtain the wrong answer. For instance,

$$x^2 + 2x - 9 = 0 \implies x^2 + 2x + 1 - 8 = 0 \implies (x + 1)^2 - 8 = 0,$$

so $x = -1 \pm \sqrt{8}$. (What was my mistake here?)
Category 1 obstacles we can address by using the resources available in the class (online notes, textbooks, office hours, classmates, Math Learning Centr), or outside resources (Khan Academy, Paul’s Online Math Notes, etc.); it is important to try to identify this sort of problem as early as possible, to maximize the time devoted to the affected topics.

Category 2 obstacles are usually addressed by practice. If I know that I can solve related rates problems, even if it takes me a while, then I can try to devote an hour to solving related rates practice problems, to see if I can pick out similarities between the questions, or even just to reinforce my ability to identify the key words of such problems.

Category 3 obstacles are often caused by aiming for speed; in multi-part problems, for instance, it is often useful to read all parts before starting to see if they share a common theme. If so, that could provide a hint as to the method of solution the instructor expects.

Finally, category 4 obstacles can only be avoided by careful proofreading, often reading through one’s work line-by-line. Sometimes we can know that we made a “figure goblin” mistake, without being able to identify it immediately: for instance, I might try to maximise the area of a square inside a circle, and despite using the correct method, wind up with an answer that is negative. These sorts of obstacles are the least important, as they affect everyone equally – it is much better to spend a few minutes to confirm that the method you have used is correct, than to spend 20 minutes reading through your work carefully to find the mix-up.