Lecture Eight

Last time we saw how some physical questions naturally give rise to Riemann sums, and hence to definite integrals. Specifically, if \( F(x) \) is the force exerted upon an object at point \( X \) over an interval \([a, b]\), then

\[
\lim_{n \to \infty} \sum_{i=1}^{n} F(x_i) \Delta x = \int_{a}^{b} F(x) \, dx.
\]

Therefore, the difficulty of solving a work problem lies in setting up the Riemann sum.

Today we will look at some related constructions. First of all, we define the average value of a function \( f(x) \) over an interval \([a, b]\). If we wish to take the arithmetic mean of certain values of \( f(x) \), say, \( f(x_1^*), f(x_2^*), \ldots, f(x_n^*) \), our formula is

\[
\text{fav} = \frac{1}{n} \left( f(x_1^*) + \cdots + f(x_n^*) \right) = \frac{1}{(b-a)} \sum_{i=1}^{n} f(x_i^*)
\]

That is,

\[
\text{fav} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx.
\]

Example: Find the average value of \( \sin x \) on the interval \([0, \pi]\).

Solution: By definition, \( \pi \)

\[
\text{fav} = \frac{1}{\pi} \int_{0}^{\pi} \sin x \, dx = \frac{1}{\pi} \left[ -\cos x \right]_{0}^{\pi} = \frac{2}{\pi}
\]

Theorem (the mean value theorem for integrals): Suppose \( f(x) \) is continuous on \([a, b]\). Then there exists \( c \in (a, b) \) such that

\[
f(c) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx.
\]

That is, \( f(x) \) always attains its average value.
proof: Let $F(x) = \int_{a}^{x} f(t) \, dt$. By FTC I we know $F(x)$ is differentiable on $(a, b)$ and continuous on $[a, b]$. Therefore, by the mean value theorem (for derivatives) there exists $c \in (a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a} \left( \int_{a}^{b} f(t) \, dt - \int_{a}^{c} f(t) \, dt \right) = \frac{1}{b - a} \int_{a}^{c} f(x) \, dx.$$ 

By FTC I we have also that

$$F'(c) = f(c),$$

and we are done.

In fact, in our example (see left), there are two points in $[0, \pi]$ which have

$$\sin c_1 = \sin c_2 = \frac{3}{\pi} = \sin \frac{\pi}{2},$$

although the theorem doesn't tell us this, or where they are.

Related to the concept of the average value is that of the centre of mass, which we now introduce.

It is a fact that, given two point masses $m_1$ and $m_2$, the system

$\begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
m_1 \\
m_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}$

is in equilibrium if (and only if) $m_1 x_1 = m_2 x_2$. In this case, the centre of mass is the fulcrum (balancing point).

If we draw this picture on the $x$-axis with the centre of mass at $\bar{x}$, we have the equation

$$m_1 (x_1 - x) = m_2 (x_2 - x) \iff m_1 x_1 + m_2 x_2 = \bar{x}(m_1 + m_2),$$

where $m_i$ is at position $x_i$. Thus, the centre of mass of two objects is defined

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2},$$

and by analogy we define the centre of mass of any $n$ objects, namely,

$$\bar{x} = \frac{m_1 x_1 + \ldots + m_n x_n}{m_1 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$
The quantity $\sum_{i=1}^{n} m_i x_i$ in the numerator is called the sum of the individual moments (with respect to the origin) of the masses $m_i$, their sum is called the moment of the system (about the origin).

We now generalize again, to two dimensions. This time we have $n$ masses $m_1, \ldots, m_n$ at co-ordinates $(x_1, y_1), \ldots, (x_n, y_n)$, and we define 2 quantities:

- the moment of the system about the $y$-axis
  \[ M_y = \sum_{i=1}^{n} m_i y_i, \]

- and the moment about the $x$-axis
  \[ M_x = \sum_{i=1}^{n} m_i x_i. \]

Notice the subscripts!

We then define the centre of mass $(\bar{x}, \bar{y})$ to be
\[ \bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}, \]
where $m = \sum_{i=1}^{n} m_i$ is the total mass of the system.

**Example:** Find the centre of mass of the system consisting of three objects at positions $(1, 1)$, $(2, -1)$, and $(3, 2)$, of respective masses $3, 4,$ and $8$.

**Solution:** Putting these values in terms of our variables gives $m_1 = 3, m_2 = 4, m_3 = 8$.

- $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2, -1)$, $(x_3, y_3) = (3, 2)$.

Thus $m = m_1 + m_2 + m_3 = 3 + 4 + 8 = 15$,

and similarly
\[ M_y = m_1 x_1 + m_2 x_2 + m_3 x_3 = (3)(-1) + (4)(2) + (8)(3) = 29, \]
\[ M_x = m_1 y_1 + m_2 y_2 + m_3 y_3 = (3)(1) + (4)(-1) + (8)(2) = 15. \]

Hence $\bar{x} = \frac{M_y}{m} = \frac{29}{15}, \quad \bar{y} = \frac{M_x}{m} = \frac{15}{15}$, and we see that the centre of mass is at
\[ (\bar{x}, \bar{y}) = \left( \frac{29}{15}, 1 \right). \]

(picture text)
These formulas naturally give rise to Riemann sums: if \( R \) is a region in the plane bounded below by \( y = 0 \), above by \( y = f(x) \), to the left by \( x = a \) and to the right by \( x = b \), of uniform density \( p \), we have for any collection of sample points \( (x_i, f(x_i)) \) that

\[
\bar{x} = \frac{1}{M} \sum_{i=1}^{N} x_i f(x_i) \Delta x_i
\]

and

\[
\overline{\Delta x} = \frac{1}{M} \sum_{i=1}^{N} \Delta x_i
\]

where \( A = \int_{a}^{b} f(x) \, dx \) is the area under the graph of \( f(x) \).

Similarly,

\[
\overline{y} = \frac{1}{M} \sum_{i=1}^{N} y_i f(x_i) \Delta x_i
\]

\[
\overline{\Delta y} = \frac{1}{M} \sum_{i=1}^{N} \Delta y_i
\]

(The factor \( \frac{1}{2} \) appears because the centre of mass of a uniform rectangle is "halfway up").

Such a region \( R \) is called a lamina, and its centre of mass

\[
(\bar{x}, \overline{\Delta x}) = \left( \frac{1}{M} \int_{a}^{b} x f(x) \, dx, \frac{1}{M} \int_{a}^{b} \frac{1}{2} f(x)^2 \, dx \right)
\]

is called the centroid.

Example: Find the centroid of a semicircle of radius \( R \) and uniform density \( \rho \).

Solution: We consider the function \( f(x) = \sqrt{R^2 - x^2} \), whose graph is the desired lamina. Since our lamina is uniformly dense, we have by our formulas above

\[
\bar{x} = \frac{1}{A} \int_{-R}^{R} x \sqrt{R^2 - x^2} \, dx,
\]

\[
\overline{\Delta x} = \frac{1}{A} \int_{-R}^{R} \frac{1}{2} \sqrt{R^2 - x^2} \, dx.
\]

We know a priori that \( A = \frac{1}{2} \pi R^2 \), and by symmetry (or the fact that the integrand is odd), we know \( \bar{x} = 0 \). Thus

\[
\overline{\Delta x} = \frac{2}{\pi R^2} \int_{-R}^{R} \frac{1}{2} (R^2 - x^2) \, dx = \frac{2}{\pi R^2} \left[ R^2 x - \frac{1}{3} x^3 \right]_{x=-R}^{x=R} = \frac{4R}{3}\pi.
\]

It follows therefore that our centroid is at \( (0, \frac{4R}{3}\pi) \).
The other situation we consider is when the region \( R \) in question is bounded above and below by curves. In this case, the formula is the same but slightly different:

If \( R \) is the region bounded above by \( y = f(x) \), below by \( y = g(x) \), to the left by \( x = a \), and to the right by \( x = b \), then its centroid is given by the formula:

\[
(x, y) = \left( \frac{1}{A} \int_a^b x[f(x) - g(x)] \, dx, \frac{1}{A} \int_a^b \frac{1}{2} [(f(x))^2 - (g(x))^2] \, dx \right)
\]

where \( A \) is as before (derivation omitted).

Example: Find the centre of mass of the region bounded by the line \( y = x \) and the parabola \( y = x^2 \).

Solution: First, we find our points of intersection to obtain the limits of integration:

\( y_1 = y_2 \iff x = x^2 \iff x(x-1) = 0 \),

and so we integrate from \( x = 0 \) to \( 1 \). Because \( x \geq x^2 \) on \([0, 1]\) (check this!), we have by our formulas above that:

\[
A = \int_0^1 (x - x^2) \, dx = \left[ \frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_{x=0}^{x=1} = \frac{1}{6}
\]

and thus:

\[
x = 6 \int_0^1 x(x-x^2) \, dx = 6 \int_0^1 x^2 - x^3 \, dx = 6 \left[ \frac{1}{3} x^3 - \frac{1}{4} x^4 \right]_{x=0}^{x=1} = \frac{1}{2}
\]

\[
y = 6 \int_0^1 \frac{1}{2} [x^2 - x^4] \, dx = 3 \left[ \frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_{x=0}^{x=1} = \frac{2}{5}
\]

Therefore, the region has centroid

\[
(x, y) = \left( \frac{1}{2}, \frac{2}{5} \right)
\]

We leave for now the discussion of “practical calculations” and “real-world scenarios” to introduce the most powerful tool we have in our integral calculus utility belt: integration by parts.
Recall from the differential calculus the product rule, which states that
\[
\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).
\]
We can rewrite this equation as
\[
\frac{d}{dx}(f(x)g(x)) = \frac{1}{dx}(f(x)g(x)) - \frac{d(f(x))g(x)}{dx},
\]
and by again forgetting that \( \frac{d}{dx} \) is NOT A FRACTION, we clear denominators to get
\[
f(x) \frac{d}{dx}(g(x)) = \frac{d(f(x)g(x))}{dx} - \frac{d(f(x))g(x)}{dx},
\]
or more compactly (with \( u = f(x) \), \( v = g(x) \)), as
\[
u \, du = d(uv) - (du) \, v.
\]
Finally, we integrate, and we get the integration by parts formula:
\[
\int udv = uv - \int v \, du.
\]

Much like \( u \)-substitution, integration by parts requires careful thought only for one singular step, and is totally mechanical otherwise.

**Example:** Find \( \int x \sin x \, dx \).

**Solution:** We have to make a choice of what is to be \( u \); like with \( u \)-substitution, we have to guess, and this is the only difficult part. Also, once we choose \( u \), the rest of the integrand must be \( dv \) (it’s been decided for us!) So if we choose \( u = x \), then \( dv \) must be \( \sin x \, dx \).

**Note the inclusion of \( dx \) in the expression for \( dv \).**

Since \( u = x \), we know \( du = 1 \), so \( du = dx \).

Similarly, \( v = \int dv = \int \sin x \, dx = -\cos x \).

Thus by our formula,
\[
\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \sin x + C.
\]
Some remarks on this example:

(1) As with u-substitution, making the right choice of \( u \) is crucial to make the technique useful.

(2) Although we "pick up" a constant \( \int \) when we integrate \( du \), we will never include it in the formula for \( u \) (i.e., we take \( C = 0 \)). We do, however, need \( C \) when we integrate the last term.

(3) There is a picture that illustrates the "meaning" of the formula geometrically, found at:

Wiki: "Integration by parts" & Visualization.