Lecture Seven

Last time we closed with a derivation of the volume of a cone. Today, we turn that construction on its head.

With the cone, we noticed that integrating the "area function" 

\[ A(x) = \pi \left( \frac{R}{H} x \right)^2 \]

where \( \frac{R}{H}x \) is the radius of the cone at the x-value x. This is the same as if we took the graph of \( y = \frac{R}{H}x \) and rotated it about the x-axis. This motivates our next topic.

Example: Find the volume of the solid obtained by rotating about the x-axis of \( f(x) = \frac{1}{2}x + 1 \), between \( x = 0 \) and 4.

Solution: As before, we are interested in the cross-sectional area. A disc of radius \( f(x) \) has area \( \pi f(x)^2 \), and so in particular for us

\[ A(x) = \pi \left( \frac{1}{2}x + 1 \right)^2 = \frac{\pi}{4} x^2 + \pi x + \pi. \]

Thus the volume we are concerned with is

\[ V = \int_{0}^{4} A(x) \, dx = \int_{0}^{4} \frac{\pi}{4} x^2 + \pi x + \pi \, dx \]

\[ = \left[ \frac{\pi}{12} x^3 + \frac{\pi}{2} x^2 + \pi x \right]_{0}^{4} = \frac{52}{3} \pi. \]

So the solid has volume \( \frac{52}{3} \pi \).

Generally speaking, the volume of any 3-dimensional object can be written

\[ V = \int A(x) \, dx, \text{ some } a, b \in \mathbb{R}, \]

for an appropriate area function \( A(x) \). In the special case of volumes of revolution of a function \( f(x) > 0 \) about the x-axis, the area function is

\[ A(x) = \pi [f(x)]^2. \]
**Proposition:** Suppose \( f(x) \geq 0 \) is integrable on \([a, b]\). Then the volume of the solid obtained by rotating the graph of \( f(x) \) about the x-axis is given by

\[
V = \int_a^b \pi [f(x)]^2 \, dx
\]

When the solid is not a volume of revolution, it can be a bit trickier.

**Example:** Find the volume of a pyramid of height \( \frac{8}{3} \) and whose base is a square of side length 12.

**Solution:** As before, we try to find the area function \( A(x) \). We look against the cross-section: this time we have a segment of the line \( y = \frac{3}{4}x \), and when this graph has height \( \frac{3}{4}x_0 \), the cross-sectional area of the square is

\[
(2 \cdot \frac{3}{4}x_0)^2 = \frac{9}{16}x_0^2 \text{; thus } A(x) = \frac{9}{16}x^2.
\]

We then finish by taking the integral:

\[
U = \int_0^\frac{8}{3} A(x) \, dx = \int_0^\frac{8}{3} \frac{9}{16}x^2 \, dx = \left[ \frac{3}{4}x^3 \right]_0^{\frac{8}{3}} = 3.84.
\]

The general approach remains the same: we want to find the area function \( A(x) \) and integrate it.

What if we want to revolve a graph about a line other than the x-axis?

**Example:** Find the volume of the solid obtained by rotating the graph of the function \( y = \frac{1}{2}x + 1 \) between \( y = 0 \) and 4.
Solution: This time we start by putting the function in terms of \( y \):
\[ y = \frac{1}{2} x + 1 \]  
\[ \Rightarrow x = 2(y - 1) = 2y - 2 \]

This gives us the distance from the \( y \)-axis to the graph of the function
\[ x(y) = 2y - 2. \]

But, because we are rotating about the line \( x = -1 \), we must add 1 to the value of \( x(y) \), as \( x = -1 \) is one unit further than the \( y \)-axis. Thus,
\[ A(y) = \pi [(2y - 2) + 1]^2 = \pi (2y - 1)^2 = \pi (4y^2 - 4y + 1), \]

and so
\[ V = \int A(y) \, dy = \int_4^1 4y^2 - 4y + 1 \, dy = \pi \left[ \frac{4}{3}y^3 - 2y^2 + y \right]_4^1 = \pi \left( \frac{13}{3} - \frac{1}{3} \right) = 5\pi. \]

In general, if we rotate about the line \( y = c \) the function \( f(x) \) between \( x = a \) and \( b \), then the volume of the resulting solid is
\[ V = \int_a^b \pi (f(x) - c)^2 \, dx. \]

Exercise: Formulate the corresponding statement for functions of \( y \) between \( y = c \) and \( d \), about \( x = k \).

Finally, we sometimes encounter the situation where the thing that we want to rotate is the region between two graphs.

Example 1: Find the volume of the solid obtained by rotating about the \( x \)-axis the region bounded by the curves \( y_1 = x \) and \( y_2 = x^2 \).
solution First, we find the limits of integration. That is, we solve
\[ y_1 = y_2 \iff x = x^2 \iff x^2 - x = 0 \iff x(x - 1) = 0, \]
\[ \text{i.e., } x = 0 \text{ or } 1. \] Thus
\[ U = \int_0^1 A(x) \, dx, \]
and we need only find the area function. It is known that
\[ y_2 = x^2 \leq x = \frac{1}{3} x \text{ for } x \in [a, b], \]
and so we know which function is “on top”; however, the cross-sectional area we obtain is not a disc, but an annulus, i.e., a washer-shaped region.

Its area is evidently the difference in areas of the “inner” and “outer” discs. Thus, in our case,
\[ A(x) = \pi y_1(x)^2 - \pi y_2(x)^2 = \pi x^2 - \pi x^4 = \pi x^2(1 - x^2), \]
\[ \text{outer area} \quad \text{inner area} \]
and we have
\[ U = \int_0^1 A(x) \, dx = \int_0^1 x^2 - x^4 \, dx = \pi \left[ \frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_{x=0}^{x=1} = \frac{2\pi}{15}. \]

To summarize: the difficulty of volume integrals lies in finding the correct area function and (sometimes) the limits of integration.

Of course, a volume integral \( U = \int A(x) \, dx \) is just an ordinary integral with particular meaning attached to the functions and variables. So too, now, we give a special name to a certain type of integral—one in which the function being integrated is force.
It is a fact (or definition?) from physics that the work done when a force $F$ is exerted over a distance $d$ is

$$W = F \cdot d = \left( \int F \cdot dx \right)$$

It is assumed $F$ is constant here. The units we use for these values are SI units (Joules, Newtons, metres, respectively, abbreviated $J$, $N$, $m$), although in many other places (e.g. in Stewart) American units may also be found. We may or may not ignore them in calculations.

Suppose the force that is applied is not constant, but instead depends on its one-dimensional position $x$. In this case, we define the work done by the integral

$$W = \int_{a}^{b} F(x) \, dx.$$  

This represents the work done by displacing an object from $x=a$ to $b$ as a force $F(x)$ is exerted upon that point $x$.

Recall: Newton's equation, $F = ma = \text{(mass)} \times \text{(acceleration)}$.

Example 4: A 100 m-long rope of uniform density is lying on the ground. If the rope weighs 10 kg, then what is the work done in lifting one end until the rope is vertical?

Solution: The rope has a linear density $\frac{\text{mass}}{\text{length}}$ of $\rho = \frac{10 \text{ kg}}{100 \text{ m}} = \frac{1}{10} \text{ kg/m}$.

So, the force acting on the vertical portion of the rope is $\rho g A$, where $h$ is the length of this portion (see sketch) and $g$ is acceleration due to gravity. Apparently, $g = 9.8 \text{ m/s}^2$.

Thus,

$$F(h) = \left( \frac{1}{10} \text{ kg/m} \right) (h \text{ m})(9.8 \text{ m/s}^2) = \frac{9.8h}{10} \text{ N}$$

and so

$$W = \int_{0}^{100} F(h) \, dh = \int_{0}^{100} \frac{9.8h}{10} \, dh = \left[ \frac{9.8h^2}{20} \right]_{0}^{100} = 4900 \text{ J}.$$
For a more detailed treatment, see p. 448 of Stewart.

Example 2: A right circular cylindrical tank of radius 1 m and height 4 m is full of water. What is the work done in pumping all the water to the top of the tank?

Solution: We approximate the work done by imagining that we pump water out as infinitely many "sheets" of water, each one a disc of radius 1 m and thickness $\Delta y$ m. This disc has mass 

$$m = \left(1000 \text{ kg/m}^3\right) \left(\Delta y \text{ m}\right) \left(\pi \left(1^2\right) \text{ m}^2\right) = 1000\pi \Delta y \text{ kg},$$

and in our Cartesian coordinate system (see image above), $y$ denotes the distance of our sheet from the top of the tank.

Thus our sheet of water must be raised $y$ metres, while a force of

$$F(y) = (1000\pi \Delta y \text{ kg}) \left(9.8 \text{ m/s}^2\right) = 9800\pi \Delta y \text{ N},$$

acts upon it. So — because the force is constant! — the work done in lifting this "sheet" is

$$\Delta W = \frac{F(y) \Delta y}{y} \approx 9800\pi \Delta y \text{ J}.$$

Taking the limit as the number of sheets goes to infinity, this "becomes"

$$W = \int_0^4 \Delta W = \int_0^4 9800\pi \Delta y \text{ J} \, dy = \int_0^4 9800\pi y \, dy = \left[4900\pi y^2\right]_0^4 = 78400\pi \text{ J},$$

or about 246,300 Joules.

Example 2: 2 Joules are required to stretch a spring from its rest length of 30 cm to a length of 42 cm. To what length will a force of 30 N stretch the spring?
solution: According to Hooke's law, the force required to hold a spring stretched x metres from its equilibrium (= rest) state is \( F(x) = kx \), where \( k \) is the spring constant. Thus

\[
W = \int_0^{0.12} F(x) \, dx = \int_0^{0.12} kx \, dx = \frac{1}{2} kx^2 \bigg|_0^{0.12} = 0.0072 \, k;
\]

thus \( k = 277.7 \), and therefore a force of 30 N satisfies

\[
30 = (277.78)x
\]

\( \Rightarrow \)

\[
x = 0.108.
\]

So the spring will stretch 10.8 cm from its rest length.