Lecture Five

Last time we saw with FTC the relationship between integration and differentiation. Specifically, we saw that to compute a definite integral, it is enough to find an anti-derivative.

By the expression $\int f(x)\,dx$, called the indefinite integral of $f(x)$, we will denote an/the anti-derivative of $f(x)$, depending on the context. For instance, $\int x\,dx = \frac{1}{2}x^2$.

"Kind of," as we saw, $\frac{1}{2}x^2 + C$, where $C \in \mathbb{R}$ is any constant, is also an anti-derivative of $f(x) = x$. Thus, we will use the following convention:

$$\int x\,dx = \frac{1}{2}x^2 + C.$$

By this expression, we represent the entire family of anti-derivatives of $f(x) = x$. Observe that, if we are computing a definite integral, it does not matter which constant $C$ we take:

$$\int_a^b x\,dx = \left[\frac{1}{2}x^2 + C\right]_a^b = \left(\frac{1}{2}b^2 + C\right) - \left(\frac{1}{2}a^2 + C\right) = \left[\frac{1}{2}x^2\right]_a^b.$$

It is crucially important to distinguish between the real number $\int_a^b f(x)\,dx$, and the function (or family of functions) $\int f(x)\,dx$; the former has limits of integration, the latter does not.

Example: Suppose $f(x) = \frac{1}{1+x^2}$; since $\frac{d}{dx}\arctan x = \frac{1}{1+x^2} = f(x)$, we know that $\int \frac{dx}{1+x^2}$ is the function $\arctan x + C$.

By FTC II, this implies that

$$\int_0^1 \frac{dy}{1+y^2} = \left[\arctan y\right]_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4},$$

a real number.
Asides: One immediate corollary of FTC, which is inexplicably given its own section in Stewart, is the identity
\[ \int_a^b F'(x) \, dx = F(b) - F(a), \]
when \( F(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\).

**u-Substitution, or the Chain Rule in Reverse**

Suppose I want to integrate \( 2x e^{x^2} \); it is not obvious just by looking what an anti-derivative might be.

However, if we remember the chain rule, we can say
\[ \int 2x e^{x^2} \, dx = \int \frac{d}{dx} (e^{x^2}) \, dx = e^{x^2} + C \quad \text{— don't forget } C! \]

Here we introduce a technique that makes this kind of integration easier to spot—namely, the \textit{u-substitution}.

We follow the same example. First, we're given a freebie integrand like \( 2x e^{x^2} \), that we don't know how to integrate. Then we introduce a new variable \( u \) and try to put our original integral solely in terms of \( u \).

How do we know what to make \( u \) equal to? Practice!

We'll have our first guess with \( u = e^{x^2} \), so \( \frac{du}{dx} = 2x e^{x^2} \), by the chain rule. Forgetting for the moment that \( \frac{du}{dx} \) is definitely not a fraction, we can "multiply by \( dx \)" to get the equation \( du = 2x e^{x^2} \, dx \), or even
\[ \frac{dx}{2x e^{x^2}} = \frac{du}{u} \]

Next, we replace \( e^{x^2} \) by \( u \) wherever we see it:
\[ \int 2x e^{x^2} \, dx = \int 2u \, du \]

Then we replace \( dx \) by \( \frac{du}{2x u} \):
\[ \int 2x e^{x^2} \, dx = \int 2u \left( \frac{du}{2x u} \right) = \int du \]

so
\[ \int 2x e^{x^2} \, dx = \int du = u + C \]

and putting it back in terms of \( x \) completes the job:
\[ \int 2x e^{x^2} \, dx = e^{x^2} + C \]
Make sense? Of course not! Let's do some examples.

Example 1: Find \( \int 4x(1+x^2) \, dx \).

Solution: We try the substitution \( u = 1 + x^2 \). Then \( \frac{du}{dx} = 2x \), so \( du = 2x \, dx \). Thus, \( dx = \frac{du}{2x} \).

Now we can write
\[
\int 4x(1+x^2) \, dx = \int 4x \, u \, dx = \int 4x \cdot u \, (\frac{du}{2x}) = \int 2u \, du.
\]

We know by FTC that \( \int u \, du = u^2 + C \), and putting it back in terms of \( x \), we get
\[
\int 4x(1+x^2) \, dx = (1+x^2)^2 + C.
\]

Example 2: Find \( \int e^y \cos(e^y) \, dy \).

Solution: Let's try \( u = e^y \), so \( \frac{du}{dy} = e^y \) and \( dy = \frac{du}{u} \).

Then
\[
\int e^y \cos(e^y) \, dy = \int \cos u \cdot \frac{du}{u} = \int \cos u \cdot du.
\]

Therefore,
\[
\int e^y \cos(e^y) \, dy = \sin(u) + C.
\]

Notice that our ability to take advantage of the substitution depends entirely on making the right choice of what to substitute.

For instance, if in example 2 we instead substitute \( u = \cos(e^y) \), \( du = -\sin(e^y) \, e^y \) (chain rule), and we get
\[
\int e^y \cos(e^y) \, dy = \int e^y \cos u \cdot \frac{du}{\sin(e^y) \, e^y} = \int \cot(e^y) \, du.
\]

In this case, it's back to the drawing board. The only real guiding principle is that \( u \) should be the "inside function" in ex. 1 we used \( \frac{d}{dx} \) \( (1+x^2)^2 \), and \( 1+x^2 \) was \( u \), and in ex. 2 we used \( \frac{d}{dy} \) \( \sin(e^y) \), and \( e^y \) was \( u \).
Example 3: Find \( \int \frac{\cos t}{1 + \sin^2 t} \, dt \).

Solution: Here we'll take \( u = \sin t \), so \( du = \cos t \, dt \).

\[
\int \frac{\cos t}{1 + \sin^2 t} \, dt = \int \frac{\cos t}{1 + u^2} \, (du) = \int \frac{du}{1 + u^2} = \arctan u + C.
\]

\( = \arctan (\sin x) + C \)

Additional exercises: Find the indefinite integrals.

(a) \( \int \frac{du}{\ln u} \)

(b) \( \int 2 \tan x \sec^2 x \, dx \)

(c) \( \int 2 \sec^2 t \tan t \, dt \)

(d) \( \int \frac{2y}{\sqrt{1 - y^4}} \, dy \)

(Try to use different substitutions for (b) and (c).)

The short version: if \( u = g(x) \) is a differentiable function whose range lies in \([a, b]\), and \( f(x) \) is continuous on \([a, b]\), then

\[
\int f(g(x)) \, g'(x) \, dx = \int f(u) \, du = F(u) + C.
\]

We saw many examples of this with indefinite integrals — how does it work with definite integrals?

Clearly, we can do the following process:
1. Take the definite integral
2. Find an anti-derivative of integrand
3. Put back in terms of original variables
4. Evaluate at limits of integration

I like to call this the "painful" method, or I like doing unnecessary work.
Example: Find \( \int_1^e \frac{dx}{x(1+\ln^2 x)} \).

Solution: We will substitute \( u = \ln x \), so \( \frac{du}{dx} = \frac{1}{x} \) and \( dx = x \cdot du \). Then

\[
\int \frac{dx}{x(1+\ln^2 x)} = \int \frac{1}{x(1+u^2)} \cdot (x \cdot du) = \int \frac{du}{1+u^2} = \arctan u + C.
\]

So the antiderivative of \( \frac{1}{x(1+\ln^2 x)} \) is \( \arctan(\ln x) + C \).

By the FTC, therefore,

\[
\int_1^e \frac{dx}{x(1+\ln^2 x)} = \left[ \arctan(\ln x) \right]_1^e
= \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.
\]

These are a better way!

Instead of doing everything (apart from the anti-

differentiation) in the “x-universe,” we’ll do everything

in the “u-universe,” in this way: when we substitute

\( u \) in the integrand, we will also substitute the

limits of integration:

\[
\int_1^e \frac{dx}{x(1+\ln^2 x)} = \int_1^e \frac{du}{1+u^2} = \left[ \arctan u \right]_1^e = \frac{\pi}{4}.
\]

Same example: if \( u = \ln(x) \), then \( u = 0 \) when \( x = 1 \) and \( u = \infty \) when \( x = e \). So,

\[
\int \frac{dx}{x(1+\ln^2 x)} = \int \frac{1}{x(1+u^2)} \cdot (x \cdot du) = \int_1^e \frac{du}{1+u^2} = \left[ \arctan u \right]_1^e = \frac{\pi}{4}.
\]

I hope it is clear which method I prefer.
example 1: Compute \( \int_0^3 2y e^{y^2} \, dy \).

Solution: We try the substitution \( t = 1 + y^2 \). Then \( \frac{dt}{dy} = 2y \), so \( dy = \frac{dt}{2y} \). We have

\[
t(0) = 1 + 0^2 = 1, \quad t(3) = 1 + 3^2 = 10,
\]

and so by our formula

\[
\int_0^3 2y e^{y^2} \, dy = \int_1^{10} e^t \left( \frac{dt}{2y} \right) = \int_1^{10} e^t \, dt = \left[ e^t \right]_1^{10} = e(10^2 - 1).
\]

\[= 100e - 2 = e(98) \]

example 2: Compute \( \int_0^{\pi/2} \cos(s \sin x) \cos x \, dx \).

Solution: We substitute \( u = \sin x \), so \( du = \cos x \, dx \). Then \( u(0) = 0 \), \( u(\pi/2) = 1 \), and so

\[
\int_0^{\pi/2} \cos(s \sin x) \cos x \, dx = \int_0^1 \cos(u) (du) = [\sin u]_0^1 = \sin 1.
\]

example 3: Find \( \int_0^4 \sqrt{2x+1} \, dx \).

Solution: We take \( u = 2x + 1 \), \( du = 2 \, dx \). Then

\[
\int_0^4 \sqrt{2x+1} \, dx = \int_u^9 u^{1/2} (du/2) = \frac{1}{2} \int_1^9 u^{1/2} \, du = \frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right]_1^9
\]

\[
= \frac{1}{3} (9^{3/2} - 1^{3/2}) = 2(3) = 6.
\]

example 4: Find \( \int_1^e \frac{\ln x}{x} \, dx \).

Solution: Take \( w = \ln x \), \( dw = \frac{1}{x} \, dx \). Then

\[
\int_1^e \frac{\ln x}{x} \, dx = \int_0^1 w \, dw = \left[ \frac{1}{2} w^2 \right]_0^1 = \frac{1}{2}.
\]