Lecture Three

Today, we start by weaving a rich tapestry of examples of sequences.

**Example 1.** Determine \( \lim_{n \to \infty} n^2 \) for \( n \in \mathbb{R} \).

**Solution:** The value of the limit will depend on \( n \).
For instance, if \( n = 0 \), then \( n^2 = 1 \) for all \( n \), and so \( \lim_{n \to \infty} n^2 = \lim_{n \to \infty} 1 = 1 \).

If \( n > 0 \), then \( f(x) = x^2 \) is an increasing function (derivative is \( f'(x) = 2x > 0 \)) on \( [1, \infty) \), positive, continuous function on \( [1, \infty) \).
By **Fact (2)** from last time, \( \lim_{n \to \infty} n^2 = \lim_{n \to \infty} X^2 = \infty \).

For \( n < 0 \), say \( n = -n \), we instead have \( \lim_{n \to \infty} n^2 = \lim_{n \to \infty} n^2 = \lim_{n \to \infty} (n^{-1})^2 \).

We saw last time that \( \lim_{n \to \infty} n^{-1} = 0 \) and since \( f(x) = x^2 \) is continuous, we have by **Fact (1)** that \( \lim_{n \to \infty} n^2 = \left( \lim_{n \to \infty} n^{-1} \right)^2 = (0)^2 = 0 \).

Thus \( \lim_{n \to \infty} n^2 = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \infty & n > 0 \end{cases} \).

**Example 2.** Determine whether or not the sequence \( a_n = \frac{n}{n^{10} + n} \) is convergent.

**Solution:** We divide the numerator and denominator by the highest power of \( n \) — in this case, \( n^1 = n \).

\[
\frac{a_n}{n^{10} + n} = \frac{n}{n^{10} + n} = \frac{1}{\frac{n}{n^{10}} + \frac{1}{n}}.
\]

The denominator tends to zero and assumes only positive values, and so its reciprocal will tend to \( \infty \).
Lemma: Let \( \{a_n\} \) be a sequence of nonzero real numbers.

(a) If \( \lim_{n \to \infty} a_n = \pm \infty \), then \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \).

(b) If \( \lim_{n \to \infty} a_n = 0 \) and only finitely many an are negative, then \( \lim_{n \to \infty} \frac{1}{a_n} = +\infty \); if instead only finitely many an have \( a_n > 0 \), then \( \lim_{n \to \infty} \frac{1}{a_n} = -\infty \).

Proof idea: If \( x_n > M > 0 \) for all \( n \geq N = N(M) \), then \( 0 < \frac{1}{x_n} < \frac{1}{M} \) for all \( n \geq N \).

Example 3: Does \( a_n = \frac{n!}{n^n} \) converge or diverge?

Solution: We observe that
\[
a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdots n}{n \cdot n \cdots n} = \left( \frac{1}{n} \right) \left( \frac{2}{n} \right) \cdots \left( \frac{n}{n} \right).
\]
Clearly, if \( 1 \leq k \leq n \) then \( 0 \leq \left( \frac{k}{n} \right) \leq 1 \), and so
\[
0 < a_n \leq \frac{1}{n} \quad \text{for all } n.
\]
Thus, since
\[
\lim_{n \to \infty} 0 = \lim_{n \to \infty} \frac{1}{n} = 0,
\]
we have by the squeeze theorem that \( \lim_{n \to \infty} a_n \) exists and equals 0.

Example 4: Fill in the table:

<table>
<thead>
<tr>
<th>( a_n )</th>
<th>monotone?</th>
<th>bounded?</th>
<th>convergent?</th>
<th>limit?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{m_n}{n} )</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>0</td>
</tr>
<tr>
<td>( -2^n )</td>
<td>yes (strictly decreasing)</td>
<td>no</td>
<td>no</td>
<td>-</td>
</tr>
<tr>
<td>( (-1)^n )</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>-</td>
</tr>
<tr>
<td>( 4 - \frac{1}{e^n} )</td>
<td>yes (strictly increasing)</td>
<td>yes</td>
<td>yes</td>
<td>4</td>
</tr>
<tr>
<td>( (-1)^n n! )</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>-</td>
</tr>
</tbody>
</table>
Remarks:

1. Every convergent sequence is bounded (why?)

2. Adding, deleting, or changing any finite number of terms in a sequence will not affect convergence or boundedness (though it may affect monotonicity).

3. If \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(n) = \frac{n}{n^2 + 1} \), then monotonicity can be indicated by the derivative:

\[
\frac{df}{dx} = \frac{1 - \ln x}{x^2}
\]

\( f'(2) > 0, \ f'(3) < 0, \) and indeed \( f'(x) < 0 \) for \( n \geq 3 \). This shows that \( a_n \) is decreasing (strictly) from \( n = 3 \) onward, but is not a decreasing sequence as \( 0 = a_1, \ a_2 = \frac{1}{2}, \ a_3 > a_4 > a_5 > \ldots \)

However, we still have \( a_3 > a_4 > a_5 > \ldots \)

Example 4: Show that \( \lim_{n \to \infty} \left( 1 + \frac{(-1)^n}{2^n} \right) = 1 \).

Solution: All limit proofs consist of three steps.

(i) Let \( \varepsilon > 0 \) — we always start with an arbitrary level of precision.

(ii) Let \( N = \ldots \) — we find the point in the sequence beyond which all terms are “\( \varepsilon \)-close” to the limiting value — in this case, 1. Observe that

\[
|1/n - 1| = \left| \frac{(-1)^n}{2^n} \right| = \frac{1}{2^n},
\]

and that

\[
\frac{1}{2^n} < \varepsilon \iff \frac{1}{2^n} < \varepsilon \iff \log\left(\frac{1}{2}\right) < n \log 2, \]

if and only if \( n > \frac{\log\left(\frac{1}{2}\right)}{\log 2} \). Thus, we are led to make \( N \) the least integer greater than \( \frac{\log\left(\frac{1}{2}\right)}{\log 2} \).

(iii) Then (blah blah blah): We put it together. If \( n \geq N \), then

\[
|1/n - 1| < \varepsilon \iff \frac{1}{2^n} < \varepsilon \iff \log\left(\frac{1}{2}\right) < n \log 2, \]

GIVEN \( \varepsilon > 0 \), we have found \( N \) such that \( n \geq N \) implies \( |1/n - 1| < \varepsilon \), hence \( \lim_{n \to \infty} 1/n = 1 \).
This ends for now our discussion on sequences, we will return in a few weeks.

Now we turn instead to the topic of integration, which will take up most of our time in this course. We start with an unfortunately unmotivated example.

Given ordinary, regular polygons, we know from kindergarten the formulae for their areas:

\[
A = \pi r^2 \\
A = \frac{1}{2}bh \\
A = lh \\
A = \frac{1}{2}(a+b)h
\]

What if we needed to find areas of irregular shapes? For instance, suppose I need to know the total area of my country estate, to reckon my property taxes; call this area \(A\), I can get upper & lower bounds on \(A\): the smaller and more numerous the rectangles I use, the closer my estimate will be.

\[
30 = R_1 + R_2 + R_3 \leq A \leq R_1 + R_2 + R_3 = 51 \\
36 = R_1 + R_2 + \ldots + R_n \leq A \leq R_1 + R_2 + \ldots + R_n = 44
\]

If I could take infinitely many rectangles and make them infinitely small, the two sides would "agree" on the true value of \(A\) (compare: squeeze theorem). Such is the idea of a Riemann sum: we approximate the area under the graph using small vertical rectangles.

(wiki: Riemann sum & animations.)
Definition: A Riemann sum is an expression of the form
\[ \sum_{i=1}^{N} f(x_i^*) (x_i - x_{i-1}) = \sum_{i=1}^{N} f(x_i^*) \Delta x_i, \]
where \( f(x) \) is a function defined on \([a, b]\), the \( x_i \) are numbers such that
\[ a = x_0 < x_1 < x_2 < \ldots < x_N = b, \]
and each \( x_i^* \) satisfies
\[ x_{i-1} \leq x_i^* \leq x_i. \]

More familiarly,
\[ \sum_{\text{intervals}} \frac{\text{height}}{\text{width}} \]

Some pictures of Riemann sums (with \( N = 4 \)):

"left", \( x_i^* = x_{i-1} \)

"middle", \( x_i^* = \frac{1}{2}(x_i + x_{i+1}) \)

"right", \( x_i^* = x_i \)

"w+g", \( x_i^* = ??? \)

Technically, Riemann sums are very general, and most will look like the fourth example. We will almost always assume that our subintervals are evenly spaced, i.e.,
\[ \Delta x_i = x_i - x_{i-1} = \frac{b-a}{N} \]
for every \( i \).
In this case, \[ x_i = x_0 + i \Delta x = x_0 + i \left( \frac{b-a}{N} \right). \]
Notice that, in our formula, the quantity $f(x^*) (x^*_i - x_{i-1})$ is negative if and only if $f(x^*) < 0$, that is, as Riemann sums approximate the area under a graph, that area below the x-axis counts as negative.

The true area under the graph of the function $f(x)$ between $a$ and $b$, which we have been calling $A$, is denoted by $\int_a^b f(x) \, dx$, called the (definite) integral of $f(x)$ from $a$ to $b$, can be approximated arbitrarily well by Riemann sums, if we have a "nice" enough function $f(x)$.

It may seem strange why we would want to solve the "area problem" in the first place, but if we can, it turns out we get a lot of other stuff for free.

Example: My Bentley's odometer is broken and my valet needs to know the distance to my country estate. It takes $\frac{1}{2}$ hr to drive there — how far is it?

Solution: I record my speed $v$ (in kph) at 5 min ($= \frac{1}{12}$ hr) intervals:

<table>
<thead>
<tr>
<th>t (min)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed</td>
<td>51</td>
<td>61</td>
<td>59</td>
<td>65</td>
<td>65</td>
<td>53</td>
<td>50</td>
</tr>
</tbody>
</table>

Since average speed is distance over time, we know that distance is $t \cdot v$. So, I make the crude assumption that, over these 5-minute intervals, my speed doesn't change. Then I can approximate

$$ d_{\text{total}} = \sum_{i=1}^N v(t_i) (t_i - t_{i-1}) $$

length of this interval

$$ v(t) = \frac{1}{12} (0.51 + 0.61 + 0.59 + 0.65 + 0.45 + 0.53 + 0.50) $$

So my butler must drive $\approx 26.5$ km. Of course, if $v(t)$ is speed in kph at time $t$ (given in hours), then what I have done is approximate $\int_{t_{12}}^{t_{30}} v(t) \, dt$ by a right Riemann sum with $N=6$ sub intervals.
Remark: In the expression $\int_a^b f(x) \, dx$, the “$dx$” term is there to indicate which variable is the variable of integration. While this only becomes important during the calculus of several variables (not this class), it is still a necessary part of the expression. The expression $\int_a^b g(x) \, dx$ is meaningless.

Returning to our notation from before, we will call $x_i$ the endpoints of our sub-intervals, and the $x_{i+1}$ will be our sample points.

Definition: The Riemann integral of a function $f(x)$ over an interval $[a, b]$ exists and equals $S$ if, for every $\varepsilon > 0$, there exists a positive integer $N$ such that

$$\left| S - \sum_{i=1}^{N} f(x_i^*) (x_i - x_{i-1}) \right| < \varepsilon$$

for any choice of sample points $x_i$. If the Riemann integral exists, it is denoted

$$\int_a^b f(x) \, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i^*) (x_i - x_{i-1})$$

and $f(x)$ itself is called integrable on $[a, b]$. If $f(x)$ is integrable, then the number $S$ is finite, and necessarily equals

$$\int_a^b f(x) \, dx$$

Fact: Every continuous function is integrable, but not every integrable function is continuous.

Properties: (1) $\int_a^b 1 \, dx = \int_a^b dx = b - a$. In particular, $\int_a^a f(x) \, dx = 0$.

(2) $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$

(3) $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$ for any $c \in \mathbb{R}$ for which this makes sense.

Useful identities: (i) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(ii) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
example 1: Compute \( \int 2x \, dx \).

solution: Let \( N > 0 \) and let \( x_i \) be evenly spaced, so \( x_i = x_0 + i \Delta x = 0 + \frac{i(1/2)}{N} = \frac{i}{N} \), and \( \Delta x = \frac{1}{N} \). We will take a right Riemann sum, so \( x_i^* = x_i \), and our Riemann sum is

\[
\sum_{i=1}^{N} f(x_i^*) \Delta x_i = \sum_{i=1}^{N} 2 \cdot \frac{i}{N^2} = \frac{2}{N^2} \sum_{i=1}^{N} i.
\]

By our useful identity, this equals

\[
\frac{2}{N^2} \left( \frac{N(N+1)}{2} \right) = \frac{N^2+N}{N^2} = 1 + \frac{1}{N}.
\]

Taking the limit as \( N \to \infty \), we obtain

\[
\int 2x \, dx = \lim_{N \to \infty} \frac{2}{N^2} \sum_{i=1}^{N} i = \lim_{N \to \infty} \left( 1 + \frac{1}{N} \right) = 1.
\]

example 2: Find \( \int_{2}^{4} g(t) \, dt \), where \( g(t) = \begin{cases} \frac{2t+3}{t-3} & \text{if } t \neq 3, \\ 7 & \text{if } t = 3. \end{cases} \)

solution: We start by using property (3) to write

\[
\int_{2}^{4} g(t) \, dt = \int_{2}^{3} g(t) \, dt + \int_{3}^{4} g(t) \, dt = \int_{2}^{3} \left( \frac{2t+3}{t-3} \right) \, dt + \int_{3}^{4} 7 \, dt.
\]

Consider first \( \int_{2}^{3} \left( \frac{2t+3}{t-3} \right) \, dt \), with \( \Delta t = \frac{b-a}{N} = \frac{1}{N} \) again, and

Then, taking again a right R.S. \( (t^*_k = t_k) \), we get

\[
\sum_{k=1}^{N} g(t_k^*) \Delta t_k = \sum_{k=1}^{N} \left( \frac{2t_k+3}{t_k-3} \right) \left( \frac{1}{N} \right) = \sum_{k=1}^{N} \frac{2(2+\frac{k}{N})+3}{3} \left( \frac{1}{N} \right) = \frac{2}{N} \sum_{k=1}^{N} \left( \frac{7+2k}{N} \right) = \frac{2}{N} \left( \frac{7}{N} + \frac{2}{N^2} \right) = \frac{2}{N} + \frac{2}{N^2}.
\]

Clearly the first sum equals 7, and the second has

\[
\frac{2}{N^2} = \frac{2}{N} \sum_{k=1}^{N} i = \frac{2}{N} \cdot \frac{N(N+1)}{2} = 1 + \frac{1}{N}.
\]

Taking the limit as \( N \to \infty \), we get

\[
\int_{2}^{3} g(t) \, dt = \lim_{N \to \infty} \sum_{i=1}^{N} g(t_i^*) \Delta t_i = \lim_{N \to \infty} \left( 7 + \left( 1 + \frac{1}{N} \right) \right) = 8.
\]

Similarly, taking \( t^*_k = t_k = 3 + \frac{1}{N} \), we find (exercise) that

\[
\int_{3}^{4} g(t) \, dt = \lim_{N \to \infty} \left( -11 - \left( 2 + \frac{2}{N} \right) \right) = -13,
\]

and hence

\[
\int_{2}^{4} g(t) \, dt = 8 + (-13) = -5.
\]

These are all the examples of Riemann sums we will do.