Lecture One

Math 101 - Integral Calculus, section 951 (Summer)
MWF M, Th, F 16.00 - 18.00 MATH bldg.
W 16.00 - 17.00 Room 100

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Recommended textbook: Early Transcendentals, Stewart (7th Ed.)
Office hours:
Evaluation: Final exam 50% (tentative date 2016/06/18)
Weekly quizzes 40% (10% each best 4 of 5)
WeBWorK 10%

Favorite colour: Red.

(Reminder about academic honesty).

Prologue - letters in mathematics.

For several years now, we have been using letters in our math classes. Broadly speaking, we have used these in two general ways. Firstly, as an unknown, i.e., a fixed quantity whose value is determined by circumstances and which may or may not be known to us.

Examples:
(i) \(167 - x^2 = 144\) \(\Rightarrow x = \pm \sqrt{23}\)
(ii) \(y = 1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} \Rightarrow y = 5050\)
(iii) \(x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{3}}{2}\).

If the value of the unknown is discovered, we may call it a constant, like our friends 

\[e = 2.718281828\cdots, \quad \pi = 3.14159265\cdots\]

We also use them as variables when they are allowed to take on any value from a given set; typically we talk about variables of functions, i.e. rules that send an element of one set (the domain) to an element of another set (the codomain or range), different (but related!) sets.
Examples: (i) \( f(x) = x^3 + x + 1 \), \( x \in \mathbb{R} \).

(ii) \( g(x) = \frac{1}{\log x} - \sin x \), \( x > 0 \).

(iii) \( h(t) = t^3 + t + 1 \), \( t \in \mathbb{Q} \).

Are examples (i) and (iii) the same function, or not?

Exercises. Determine the domain and range of each of these functions.

Let \( \mathbb{N} = \{1, 2, 3, \ldots\} \) denote the set of positive integers, and \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \) the non-negative ones. By a sequence of real numbers, we mean a list of real numbers indexed by \( \mathbb{N} \) (or \( \mathbb{N}_0 \)).

The sequence differs from the set in that

1. the set is unordered, while the sequence is ordered.
2. the set has no repetition, while the sequence may.
3. given a sequence
   \[ (a_1, a_2, a_3, \ldots) \]
we may obtain the "set of points"
\[ \{a_1, a_2, a_3, \ldots\} \] (ignoring repetition)

but given a set we cannot generally form a sequence.

Notation: We will use the notational conventions:
\[ (a_n) = (a_1, a_2, a_3, \ldots) \]

One also commonly sees
\[ \{a_n\}, \{a_n\}_{n=1}^{\infty}, (a_n)_{n=1}^{\infty} \]
but we caution against confusing the set \( \{a_n\} \) (which has one element, the \( n \)-th term of the sequence) with the sequence \( (a_n) = (a_1, a_2, a_3, \ldots) \).

Examples: (1) The sequence \( (1, 2, 3, 4, 5, \ldots) \) whose \( n \)-th term is \( n \).

(2) The constant sequence \( (1, 1, 1, 1, \ldots) \).

(3) The "arithmetic" sequence \( (1, 4, 7, 10, 13, \ldots) \).

(4) The "geometric" sequence \( (3, 12, 48, 144, 432, \ldots) \).

(5) The Fibonacci sequence \( (0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots) \).

(6) The "alternating" sequence \( (-1, 1, -1, 1, -1, \ldots) \).

(7) The sequence \( (2, 3, 5, 7, 11, 13, 17, \ldots) \) of primes.

(8) The Kolakoski sequence \( (1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 1, \ldots) \).
There are many ways to describe a sequence. One way is to list terms, as we have done. Another way is to define it recursively, by stating the first (few) term(s) and giving the rest in terms of these.

For the same examples:
(1) $a_1 = 1$; $a_{n+1} = a_n + 1$ for $n \geq 2$.
(2) $a_1 = 1$; $a_{n+1} = a_n$ for $n \geq 2$.
(3) $a_1 = 1$; $a_{n+1} = a_n + 3$ for $n \geq 2$.
(4) $a_1 = 3$; $a_{n+1} = 4 \cdot a_n$ for $n \geq 2$.
(5) $a_1 = 0$; $a_2 = 1$; $a_{n+1} = a_n + a_{n-2}$ for $n \geq 3$.
(6) $a_1 = -1$; $a_{n+1} = -a_n$ for $n \geq 2$.

Sometimes we can give a closed form expression for $(a_n)$, that is, we can write down a function $f: \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$.

Some examples:
(1) $a_n = n$
(2) $a_n = 1$
(3) $a_n = 3n - 2$
(4) $a_n = 4^{n-1} \cdot 3$
(5) $a_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$ where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$
(6) $a_n = (-1)^n$

Exercise: try to deduce these closed form formulas from the recursive formulas.

Remark: Every sequence $(a_n)$ can, in fact, be considered a function $a(n)$ whose domain is $\mathbb{N}$ and which satisfies $a(n) = a_n$; it is not however, always easy to write the formula down.

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Graph of $a(n) = \frac{1}{n}$

Plot of $a_n$ on the number line
The usual arithmetic operations $+,-,\times,\div$ extend naturally to sequences by componentwise operations: e.g.,
\[
(a_n) + (b_n) = (a_1, a_2, a_3, a_4, \ldots) + (b_1, b_2, b_3, b_4, \ldots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots),
\]
\[
(a_n) \cdot (b_n) = (a_1 b_1, a_2 b_2, a_3 b_3, \ldots).
\]

Note that we can only "divide" when the "denominator" has no zero terms:
\[
(a_n) \div (b_n) = \left( \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots \right).
\]

**Example:** Suppose $(a_n) = (n+2)$ and $(b_n) = (\frac{1}{5^n})$. Then
\[
(a_n) \cdot (b_n) = \left( \frac{n+2}{5^n} \right) = \left( \frac{3}{5}, \frac{4}{25}, \frac{5}{125}, \frac{6}{625}, \frac{7}{3125}, \ldots \right).
\]

**Example:** Suppose $(c_n)$ is the sequence of daily costs of a factory in $\$ (on day $n$), while $(r_n)$ is the sequence of daily revenues. The sequence of daily profits is then given by
\[
(p_n) = (r_n) - (c_n) = (r_n - c_n).
\]

When talking about a sequence there are certain properties we may want to consider. For example, if our sequence $(a_n)$ satisfies the inequalities
\[
a_1 \leq a_2 \leq a_3 \leq a_4 \leq \ldots
\]
then it is said to be increasing. Similarly, if
\[
a_1 > a_2 > a_3 > a_4 > \ldots
\]
then $(a_n)$ is said to be decreasing. We use the qualities strictly when the inequalities are sharp, i.e.,
\[
a_1 < a_2 < a_3 < \ldots
\]
is strictly increasing while
\[
a_1 > a_2 > a_3 > \ldots
\]
is strictly decreasing.

**Question:** When is a sequence both increasing and decreasing?
We might also ask whether the terms of our sequence stay small, or if they can become as big as we want (i.e. arbitrarily large).

**Definition:** A sequence \((a_n)\) is bounded above if there exists a number \(M \in \mathbb{R}\) such that \(a_n \leq M\) for all \(n \in \mathbb{N}\). Similarly, \((a_n)\) is bounded below if there is some \(m \in \mathbb{R}\) satisfying \(m \leq a_n\) \(\forall n \in \mathbb{N}\) (\(\forall = \) “for all”).

If \((a_n)\) is both bounded above and bounded below, it is called bounded.

Roughly speaking: a sequence is bounded if, when we plot it on the number line, it takes up only an interval of finite length, or when we graph it, the y-values of its points are all within some finite range.

**Fact:** A sequence \((a_n)\) is bounded if and only if there exists some \(K > 0\) such that \(|a_n| < K\) \(\forall n \in \mathbb{N}\), i.e., \(-K < a_n < K\) for all \(n \in \mathbb{N}\).

**Proof:** (Exercise)

**Examples (from above):**
1. \((a)\) is (strictly) increasing.
2. \((b)\) is increasing and decreasing.
3. \((c)\) is (strictly) increasing.
4. \((d)\) is (strictly) increasing.
5. \((e)\) is increasing (not strictly!)
6. \((f)\) is neither increasing nor decreasing.

**Same examples:**
1. \((a)\) is bounded below.
2. \((b)\) is bounded.
3. \((c)\) is bounded below.
4. \((d)\) is bounded below.
5. \((e)\) is bounded below.
6. \((f)\) is bounded.

An example of a “totally unbounded” sequence is \((a_n) = (-1)^n n = (-1, 2, -3, 4, -5, 6, \ldots)\).
Very often, we are interested not only in the sequence itself, but also in any values it might be “approaching.” An example: Suppose a ball is manufactured which on every bounce, attains exactly 60% of the height it was dropped from.

If \( a_n \) is the ball’s height before the \( n \)th bounce, we see

\[
(a_n) = (1, 0.6, 0.6^2, 0.6^3, \ldots)
\]

Clearly \( (a_n) \) is a decreasing sequence of positive real numbers.

Real-world experience (dangerous in math!) tells us that the ball should stop bouncing “eventually.” How can we make this notion more rigorous?

First of all, we notice that, given any height \( h \) (of \( h \) metres, say), only finitely many of our bounces can possibly be higher than \( h \). Indeed,

\[
(0.6)^n > h \iff n \log(0.6) > \log h 
\]

\[
\iff n > \frac{\log h}{\log(0.6)}
\]

(the inequality in the last line flips because we divide by \( \log(0.6) \), which is negative). That is: given any arbitrary distance \( h > 0 \), only finitely many terms of our sequence are \( \epsilon h \) away from zero.

This motivates our definition, to be seen in the next lecture.
More examples:

1. Let $s_n$ be the number of cells obtained by mutually joining each of $n$ distinct points on the circumference of a circle.

   - $n=1, s_1 = 1$
   - $n=2, s_2 = 2$
   - $n=3, s_3 = 2^{n-1}$ (however)
   - $n=4, s_4 = 2^3$
   - $n=5, s_5 = 2^4$
   - $n=6, s_6 = 31$
   - $n=7, s_7 = 57$

2. It is important to verify any guess of a formula for a sequence; patterns do not always continue in the most obvious way.

   - (a) Is this sequence bounded? monotone?

(2) Let $e_n = (1 + \frac{1}{n})^n$; we compute the first few terms:

   - $e_1 = 2, e_2 = (\frac{3}{2})^2 = 2.25, e_3 = (\frac{4}{3})^3 = 2.370$

By the binomial theorem,

$$e_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k}$$

Just by looking, it is difficult to say much about this sequence, i.e., whether it is increasing, bounded, etc. Even investigating the related function does not help:

$$y = (1 + \frac{1}{x}) \Rightarrow \log y = x \log(1 + \frac{1}{x})$$

$$\Rightarrow \frac{dy}{y} = \log(1 + \frac{1}{x}) + x \left(\frac{1}{1+x} \cdot \frac{-1}{x^2}\right) = \log \left(1 + \frac{1}{x}\right) - \frac{1}{1+x}$$
We will return to the sequence \( (a_n) \) in due course.

3. Consider the sequence
\[
x_n = \sum_{i=1}^{n} \frac{1}{i^n}
\]
if \( n \) is odd,
if \( n \) is even.

This sequence neither approaches a value, nor diverges to infinity. However, it is clear that the subsequence \( (y_n) = (x_{2n-1}) \) diverges to infinity, and that the subsequence \( (z_n) = (x_{2n}) \) of even terms converges to zero.

4. The decimal expansions of real numbers naturally give rise to several sequences. For instance, let \( (d_n) \) be the sequence whose \( n \)th term is the truncation of \( \sqrt{2} \) at the \( n \)th decimal place, so
\[
d_1 = 1.4, \quad d_2 = 1.41, \quad d_3 = 1.414, \quad d_4 = 1.4142,
\]
and so on. Notice that
(i) each \( d_n \) is a rational number, but \( \sqrt{2} \) is not.
(ii) the sequence is (not necessarily) increasing, and is bounded (what by? below? above?).
(iii) if \( p_n \) denotes the \( n \)th digit to the right of the decimal place in the decimal expansion of \( \sqrt{2} \), then we can write
\[
d_n = 1 + \sum_{k=1}^{n} p_k \cdot 10^{-k}.
\]

5. Building on (4), given any sequence \( \{r_n\} \) of numbers in the set \( \{0,1,2,\ldots,9\} \), we can construct a new sequence
\[
R_n = \sum_{k=1}^{n} r_k \cdot 10^{-k}.
\]
For instance, taking \( R_n = 9 \) for every \( n \) gives us
\[
R_1 = 0.9, \quad R_2 = 0.99, \quad R_3 = 0.999, \ldots, \quad R_n = 0.99\ldots9\quad\text{for } n \text{ copies}
\]