Lecture Thirty-Four

A survey of examples.

**Example 1.** Compute \( \int \frac{\tan^3 x}{\cos^3 x} \, dx \).

**Solution:** First, we rewrite the integral as \( \int \tan^3 x \sec^3 x \, dx \).

Recognizing the trig integral, we write
\[
\tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \sec x \tan x \, dx,
\]
then substitute \( u = \sec x \), \( du = \sec x \tan x \, dx \), then \( \tan^2 x = \sec^2 x - 1 \):
\[
\int \tan^3 x \sec^3 x \, dx = \int (u^2 - 1) u^2 \, du,
\]
and the integral is now easy. Alternatively, writing \( \tan x = \frac{\sin x}{\cos x} \) gives
\[
\int \tan^3 x \, dx = \int \sin^3 x \cos^{-3} x \, dx = \int \frac{1 - \cos^2 x}{\cos x} \sin x \, dx,
\]
and substituting \( u = \cos x \), \( du = -\sin x \, dx \) gives
\[
\int \tan^3 x \, dx = -\int \frac{1 - u^2}{u^2} \, du.
\]

**Example 2.** Find \( \int \frac{x^5 + 1}{x^3 - 3x^2 - 10x} \, dx \).

**Solution:** We notice that the denominator factors completely:
\[
x^3 - 3x^2 - 10x = x(x^2 - 3x - 10) = x(x - 5)(x + 2).
\]
So, we should use partial fractions. First, we long-divide:
\[
\frac{x^2 + 3x + 19}{x^3 - 3x^2 - 10x} = \frac{x^2 + 3x + 19}{x(x - 5)(x + 2)}.
\]

We express the integrand as:
\[
\frac{x^2 + 3x + 19}{x^3 - 3x^2 - 10x} = \frac{A}{x} + \frac{B}{x - 5} + \frac{C}{x + 2}.
\]

Multiplying both sides of the equation by \( x(x - 5)(x + 2) \) gives
\[
x^2 + 3x + 19 = A(x - 5)(x + 2) + Bx(x + 2) + Cx(x - 5).
\]

Evaluating at specific values of \( x \) to solve for \( A, B, \) and \( C \):
\[
\begin{align*}
A &= \frac{19}{3} - \frac{10}{2} = 3 \quad \text{(evaluated at } x = 0)\
B &= \frac{19 - 10}{5} = 1.8 \quad \text{(evaluated at } x = 5)\
C &= \frac{19 - 10}{2} = 4.5 \quad \text{(evaluated at } x = -2)\
\end{align*}
\]

Thus, we have
\[
\frac{x^2 + 3x + 19}{x^3 - 3x^2 - 10x} = \frac{3}{x} + \frac{1.8}{x - 5} + \frac{4.5}{x + 2}.
\]

And then we find the p.f.d. of
\[
\frac{87x^2 + 190x + 1}{x^3 - 3x^2 - 10x}.
\]
example 3: Find \( \int \frac{dx}{x \sqrt{\ln x}} \)

Solution: A simple change of perspective gives us
\[ \int \frac{dx}{x \sqrt{\ln x}} = \int \frac{dx}{x \ln x} = \int \frac{1}{u} du, \]
where \( u = \ln x, \) \( du = \frac{dx}{x}. \) The integral is now easy!

example 4: Find \( \int \sqrt{\frac{1-x}{1+x}} \, dx. \)

Solution: We'll tackle this one by writing \( \sqrt{\frac{1-x}{1+x}} = \frac{\sqrt{1-x}}{\sqrt{1+x}}, \) then multiply the denominator by its conjugate:
\[ \int \frac{1-x}{\sqrt{1+x}} \, dx = \int \frac{(1-x)^2}{\sqrt{1+x} \left( \sqrt{1+x} \right)^2} \, dx = \int \frac{1-x}{\sqrt{1-x^2}} \, dx \pm \int \frac{x}{\sqrt{1-x^2}} \, dx. \]

We know the first integral, and the second is an easy substitution.

examples: Find \( \int \arctan nx \, dx. \)

Solution: Most options of our flowchart don't apply, but we know how to differentiate the integrand. So, we integrate by parts, with \( u = \arctan nx, \) \( dv = dx. \) Then
\[ du = \frac{1}{1+n^2 x^2} \, dx \text{, } v = x. \]

Thus
\[ \int \arctan nx \, dx = x \arctan nx - \int \frac{x}{1+n^2 x^2} \, dx. \]

We write this last integral
\[ \int \frac{x}{1+n^2 x^2} \, dx = \int \frac{x}{1+x^2} \left( \frac{dx}{2 \sqrt{x^2}} \right) = \int \frac{u^2}{1+u^2} \, du, \]
where \( u = \sqrt{x}, \) \( du = \frac{dx}{2 \sqrt{x}} \), and compute \( \int \frac{u^2}{1+u^2} \, du = \int \left( 1 - \frac{1}{1+u^2} \right) \, du \).
example 6. Find $\int \tan^2 \theta \, d\theta$.

Solution. If we use the identity $\tan^2 \theta = \sec^2 \theta - 1$, we have

$$\int \tan^2 \theta \, d\theta = \int (\sec^2 \theta - 1) \, d\theta.$$ 

To compute $\int \sec^2 \theta \, d\theta$, we note that $\sec^2 \theta$ is easily differentiable and $\sec^2 \theta$ is easily integrable, so we integrate by parts:

$$u = \theta, \quad dv = \sec^2 \theta \, d\theta \Rightarrow du = d\theta, \quad v = \tan \theta$$

$$\Rightarrow \int \sec^2 \theta \, d\theta = \tan \theta - \theta + C.$$ 

and this last integral is one we just got to remember.

example 7. A tank of dimensions $2m \times 1m \times 1m$ is full of water. Find the work needed to pump half of the water out the top of the tank.

Solution. We approximate the work by imagining that we pump water out as finitely many "sheets" of water, of length $2m$, width $1m$, and depth $\Delta x$ m.

This has mass

$$\text{mass} = (2 \times 1 \times \Delta x) \text{m}^3 \cdot (1000 \text{kg/m}^3) = 2000 \Delta x \text{ kg}.$$ 

If the sheet of water is $x$ m below the top of the tank, we must move it $x$ m to pump it out of the tank. Thus the work done moving this sheet is

$$W_i = \left(2000 \Delta x \text{ kg} \right) \left(9.8 \text{ m/s}^2 \right) \left(x \text{ m} \right) = 19,600 \Delta x \text{ J}.$$ 

Taking the limit as the number of sheets goes to infinity, this becomes $19,600 \times \int_0^1 \Delta x \, dx$. Since $x$ varies between 0 and 1, the total work done is therefore

$$W = \int_0^1 19,600 \times dx \left(\approx \sum_{i=1}^{\infty} W_i \right)$$

$$= \left[9800 \times x^2 \right]_0^1 = 2450 \text{ J}.$$ 

example 8. 2 Joules are required to stretch a spring from its rest length of 130 cm to a length of 42 cm. To what length will a force of 50 N stretch the spring?
4) solution. According to Hooke's law, the force required to hold the spring stretched $x$ metres beyond its natural length is $F(x) = kx$. Thus
\[
\int_{0}^{0.12} kx \, dx = 25 = \left[ \frac{1}{2} kx^2 \right]_{0}^{0.12} = k \left( \frac{1}{2} (0.12)^2 - \frac{1}{2} (0.0)^2 \right) = 0.0072k
\]
Thus $k = 277.8$, and therefore a force of 30 N satisfies
\[
30 = (277.8) x 
\]
$\Rightarrow x = 0.108$
So the spring will stretch 10.8 cm from its rest length.

Example 9. Find the moments and centre of mass of the system of masses $m_1 = 3$ kg, $m_2 = 4$ kg, $m_3 = 8$ kg, at respective points $(-1, 1)$, $(2, -1)$, $(3, 2)$.

Solution: By our formula, we know the moment about the $y$-axis is
\[
M_y = m_1 x_1 + m_2 x_2 + m_3 x_3 = (3)(-1) + (4)(2) + (8)(3) = 29
\]
and similarly the moment about the $x$-axis is
\[
M_x = m_1 y_1 + m_2 y_2 + m_3 y_3 = (3)(1) + (4)(-1) + (8)(2) = 15
\]
It follows that
\[
\bar{x} = \frac{M_y}{m} = \frac{29}{15}, \quad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1
\]
since $m = m_1 + m_2 + m_3 = 15$.

Example 10. Find the centroid of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, $x = \frac{\pi}{4}$.

Solution: We refer to our formulas; we have
\[
A = \int_{0}^{\frac{\pi}{4}} \left( \cos x - \sin x \right) \, dx = \left[ \sin x + \cos x \right]_{0}^{\frac{\pi}{4}} = \sqrt{2} - 1
\]
(since $\cos x > \sin x$ on $[0, \frac{\pi}{4}]$). Thus
\[
\bar{x} = \frac{1}{\sqrt{2} - 1} \int_{0}^{\frac{\pi}{4}} x(\cos x - \sin x) \, dx = \left( \text{int. by parts} \right) \frac{1}{\sqrt{2} - 1} \left[ (x \sin x + \cos x) - (-x \cos x + \sin x) \right]_{0}^{\frac{\pi}{4}}
\]
\[
= \frac{\sqrt{2} \pi - 4}{4}
\]
example 11: Find the value of \( \sum_{n=1}^{\infty} \frac{9^n + 2^n}{10^n} \).

solution: First we split up the summand \( \frac{9^n + 2^n}{10^n} = \frac{9^n}{10^n} + \frac{2^n}{10^n} \).

Since \( \sum_{n=1}^{\infty} \frac{9^n}{10^n} \), \( \sum_{n=1}^{\infty} \frac{2^n}{10^n} \) are both convergent geometric series, we know
\[
\sum_{n=1}^{\infty} \left( \frac{9^n}{10^n} + \frac{2^n}{10^n} \right) = \sum_{n=1}^{\infty} \left( \frac{9}{10} \right)^n + \sum_{n=1}^{\infty} \left( \frac{2}{10} \right)^n
\]

We know \( \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \) for \( |r| < 1 \), so \( \sum_{n=1}^{\infty} \frac{9^n}{10^n} = \frac{1}{1-\frac{9}{10}} - 1 = 10 - 1 = 9 \).

Thus \( \sum_{n=1}^{\infty} \frac{2^n}{10^n} = \left( \frac{\frac{2}{10}}{1-\frac{2}{10}} \right) = \left( \frac{\frac{1}{5}}{1-\frac{1}{5}} \right) = 9 + \frac{1}{4} = \frac{37}{4} \).

\[
\sum_{n=1}^{\infty} \frac{9^n + 2^n}{10^n} = \frac{9}{1-\frac{9}{10}} + \frac{\frac{1}{4}}{1-\frac{1}{5}} = \frac{37}{4}.
\]

example 12: Determine whether or not the series
\[
\sum_{n=1}^{\infty} \frac{(-1)^n n^3 2^n}{n!}
\]
converges.

solution: We could try to find out if this satisfies the hypotheses of the alternating series test, but that may be difficult (e.g., we can't differentiate it if we replace \( n \) with \( x \)). This one is easier if we use the ratio test:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^n (n+1)^3 2^{n+1}}{(n+1)!} \cdot \frac{n^3 2^n}{n!} \right| = \lim_{n \to \infty} \frac{(n+1)^3 2^{n+1} n^3 2^n}{(n+1)! n!} = \lim_{n \to \infty} \left( \frac{(n+1)^3}{(n+1)^3} \right) \cdot \frac{2}{n+1} \to 0.
\]

Since the limit exists and is less than one, we know that the series converges.

example 13: Determine the convergence of
\[
\sum_{n=1}^{\infty} \frac{(-1)^n n^3 + 1}{n^3 - 7}.
\]
solution: Our first tool in checking convergence should always be the divergence test, since it’s fairly quick and easy. Here we see
\[ \lim_{n \to \infty} 1_{an1} = \lim_{n \to \infty} \left| \frac{n^3 + 1}{n^3 - 7} \right| = 1 \neq 0, \]
so the series must diverge.

example 14: Determine the convergence of
\[ \sum_{n=1}^{\infty} \frac{(-1)^n \cos n}{n^5}. \]

Solution: This one is sort of tricky since it is not obviously divergent (as in ex. 13) nor is it alternating, nor are all its terms positive. Even the ratio test fails, since \[ \lim_{n \to \infty} \frac{\cos(n+1)}{\cos n} \]
does not exist.

However, if we can show that it is absolutely convergent, this will be enough! And indeed, we have
\[ 1_{an1} = \left| \frac{(-1)^n \cos n}{n^5} \right| \leq \left| \frac{1}{n^5} \right|, \]
Thus since \[ \sum_{n=1}^{\infty} \frac{1}{n^5} \]
is convergent by the p-test, we know by the comparison test that \[ \sum_{n=1}^{\infty} \frac{(-1)^n \cos n}{n^5} \]
is absolutely convergent, and thus convergent.

example 15: Evaluate \[ \int \frac{\arctan x}{x} \, dx \]
as a power series.

solution: We know the power series of \[ \arctan x \]
about 0 is
\[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \]
valid on \([-1, 1]\).

Thus \[ \frac{\arctan x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1}, \]
and so on \((-1, 1)\) we have
\[ \int \frac{\arctan x}{x} \, dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{(2n+1)^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2(n+1)^2}. \]

As before, we know the radius of convergence is 1 and by checking endpoints (e.g. \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)^2} \] converges by alternating series test), we know the interval of convergence is \([-1, 1]\).
Example 16: Find the Taylor series of $e^{-x}$ about $0$.

Solution: We know the Maclaurin series of $e^x$ is
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{valid on } (-\infty, \infty), \]
So we have that $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$, and similarly
\[ xe^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!} = x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \cdots \]
If we like, we can check its radius of convergence with the ratio test:
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x|}{(n+1)!} \frac{n!}{|x|^n} = \lim_{n \to \infty} \frac{1}{n+1} = 0, \quad \text{for any } x \in \mathbb{R}. \]
So the series converges everywhere.

Example 17: Use series to evaluate
\[ \lim_{x \to 0} \frac{\sin x - x + \frac{1}{6} x^3}{x^5}. \]

Solution: The Maclaurin series for $\sin x$ is
\[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \text{valid for all } x \in \mathbb{R}, \]
and so
\[ \frac{\sin x - x + \frac{1}{6} x^3}{x^5} = \frac{1}{x^5} \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) - 1 + \frac{1}{6} x^2 = \frac{1}{x^5} \left( \frac{x^3}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots \right) = \frac{1}{x^5} \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n-4}}{(2n+1)!}. \]
That is,
\[ \frac{\sin x - x + \frac{1}{6} x^3}{x^5} = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n-4}}{(2n+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2}}{(2m+5)!}. \]
and thus
\[ \lim_{x \to 0} \frac{\sin x - x + \frac{1}{6} x^3}{x^5} = \lim_{x \to 0} \frac{1}{x^5} \left( \frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} + \cdots \right) = \frac{1}{5!} = \frac{1}{120}. \]