Super-rigid Donaldson-Thomas Invariants

Kai Behrend and Jim Bryan

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Abstract

We solve the part of the Donaldson-Thomas theory of Calabi-Yau threefolds which comes from super-rigid rational curves. As an application, we prove a version of the conjectural Gromov-Witten/Donaldson-Thomas correspondence of [MNOP] for contributions from super-rigid rational curves. In particular, we prove the full GW/DT correspondence for the quintic threefold in degrees one and two.

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1 Introduction

Let $Y$ be a smooth complex projective Calabi-Yau threefold. Let $I_n(Y, \beta)$ be the moduli space of ideal sheaves $I_Z \subset \mathcal{O}_Y$, where the associated subscheme $Z$ has maximal dimension equal to one, the holomorphic Euler characteristic $\chi(\mathcal{O}_Z)$ is equal to $n$, and the associated 1-cycle has class $\beta \in H_2(Y)$.

Recall that $I_n(Y, \beta)$ has a natural symmetric obstruction theory \cite{Th00}, \cite{BF05}. Hence we have the (degree zero) virtual fundamental class of $I_n(Y, \beta)$, whose degree $N_n(Y, \beta) \in \mathbb{Z}$ is the associated Donaldson-Thomas invariant.

Let $C = \sum_{i=1}^s d_i C_i$ be an effective cycle on $Y$, and assume that the $C_i$ are pairwise disjoint, smoothly embedded rational curves with normal bundle $N_{C_i}/Y \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Such curves are called super-rigid rational curves in $Y$ \cite{Pa99, BP01}. Assume that the class of $C$ is $\beta$.

Let $J_n(Y, C) \subset I_n(Y, \beta)$ be the locus corresponding to subschemes $Z \subset Y$ whose associated cycle under the Hilbert-Chow morphism is equal to $C$ (see Definition 2.1). Since $J_n(Y, C) \subset I_n(Y, \beta)$ is open and closed (see Remark 2.2), we get an induced virtual fundamental class on $J_n(Y, C)$ by restriction. We call $N_n(Y, C) = \deg[J_n(Y, C)]^{vir}$ the contribution of $C$ to the Donaldson-Thomas invariant $N_n(Y, \beta)$.

The goal of this paper is to compute the invariants $N_n(Y, C)$.

To formulate our results, we define a series $P_d(q) \in \mathbb{Z}[[q]]$, for all integers $d \geq 0$ by

$$\prod_{m=1}^{\infty} (1 + q^m v)^m = \sum_{d=0}^{\infty} P_d(q) v^d. \quad (1)$$

Moreover, recall the McMahon function

$$M(q) = \prod_{m=1}^{\infty} \frac{1}{(1 - q^m)^m}. \quad (2)$$

Then we prove (Theorem 2.14) that

$$\sum_{n=0}^{\infty} N_n(Y, C) q^n = M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q).$$

Maulik, Nekrasov, Okounkov, and Pandharipande have conjectured a beautiful correspondence between Gromov-Witten theory and Donaldson-Thomas theory which we call the GW/DT correspondence.

As an application of the above formula we prove the GW/DT correspondence for the contributions from super-rigid rational curves (Theorem 3.1). In particular, we prove the full degree $\beta$ GW/DT correspondence (Conjecture 3 of [MNOP]) for any $\beta$ for which it is known that all
cycle representatives are supported on super-rigid rational curves (Corollary 3.2). For example, our results yield the GW/DT correspondence for the quintic threefold in degrees one and two (Corollary 3.3). As far as we know, these are the first instances of the GW/DT conjecture to be proved for compact Calabi-Yau threefolds.

The local GW/DT correspondence for super-rigid rational curves follows from the results of [MNOP] as a special case of the correspondence for toric Calabi-Yau threefolds. In contrast to Gromov-Witten theory, passing from the local invariants of super-rigid curves to global invariants is non-trivial in Donaldson-Thomas theory, and can be regarded as the main contribution of this paper.

1.1 Weighted Euler characteristics

Our main tool will be the weighted Euler characteristics introduced in [Be05]. Every scheme $X$ has a canonical $\mathbb{Z}$-valued constructible function $\nu_X$ on it. The weighted Euler characteristic $\chi(X)$ of $X$ is defined as

$$\chi(X) = \chi(X, \nu_X) = \sum_{n \in \mathbb{Z}} n \chi(\nu_X^{-1}(n)).$$

More generally, we use relative weighted Euler characteristics $\chi(Z, X)$ defined as

$$\chi(Z, X) = \chi(Z, f^*\nu_X),$$

for any morphism $f: Z \to X$. Three fundamental properties are

(i) if $X \to Y$ is étale, then $\chi(Z, X) = \chi(Z, Y),$

(ii) if $Z = Z_1 \sqcup Z_2$ is a disjoint union, $\chi(Z, X) = \chi(Z_1, X) + \chi(Z_2, X),$

(iii) $\chi(Z_1 \times X_1, Z_2 \times X_2) = \chi(Z_1 \times Z_2, X_1 \times X_2).$

The main result of [Be05], Theorem 4.18, asserts that if $X$ is a projective scheme with a symmetric obstruction theory on it, then

$$\deg [X]^{vir} = \chi(X).$$

Thus we can calculate $N_n(Y, C)$ as $\chi(J_n(Y, C)).$

We will also need the following fact. If $X$ is an affine scheme with an action of an algebraic torus $T$ and an isolated fixed point $p \in X$, and $X$ admits a symmetric obstruction theory compatible with the $T$-action, then

$$\nu_X(p) = (-1)^{\dim T_p X},$$

where $T_p X$ is the Zariski tangent space of $X$ at $p$. This is the main technical result of [BF05], Theorem 3.4.

Finally, we will use the following result from [BF05]. If $X$ is a smooth threefold (not necessarily proper), then

$$\sum_{m=0}^{\infty} \chi(\operatorname{Hilb}^m X) q^m = M(-q)^{\chi(X)}.$$

In the case where $X$ is projective and Calabi-Yau, the above proves Conjecture 1 of [MNOP]. Note that Conjecture 1 of [MNOP] has also been proved in [Li06] and [LP06].
2 The Calculation

2.1 The open subscheme $J_n(Y, C)$

Definition 2.1 Let $C_1, \ldots, C_s$ be pairwise distinct, super-rigid rational curves on $Y$ and let $(d_1, \ldots, d_s)$ be an $s$-tuple of non-negative integers. Let $C = \sum_i d_i C_i$ be the associated 1-cycle on $Y$ and let $\beta$ be the class of $C$ in homology. Define

$$J_n(Y, C) \subset I_n(Y, \beta)$$

to be the open and closed subscheme consisting of subschemes $Z \subset Y$ whose associated 1-cycle is equal to $C$.

Remark 2.2 To see that $J_n(Y, C)$ is, indeed, open and closed, consider the Hilbert-Chow morphism, see [Ko96], Chapter I, Theorem 6.3, which is a morphism

$$f : I_n(Y, \beta)^{\text{sn}} \longrightarrow \text{Chow}(Y, d),$$

where Chow$(Y, d)$ is the Chow scheme of 1-dimensional cycles of degree $d = \deg \beta$ on $Y$. It is a projective scheme. Moreover, $I_n(Y, \beta)^{\text{sn}}$ is the semi-normalization of $I_n(Y, \beta)$. The structure morphism $I_n(Y, \beta)^{\text{sn}} \rightarrow I_n(Y, \beta)$ is a homeomorphism of underlying Zariski topological spaces. Therefore the Hilbert-Chow morphism descends to a continuous map of Zariski topological spaces

$$|f| : |I_n(Y, \beta)| \longrightarrow |\text{Chow}(Y, d)|.$$  

Because the $C_i$ are super-rigid, the cycle $C$ corresponds to an isolated point of $|\text{Chow}(Y, d)|$. So the preimage of this point under $|f|$ is open and closed in $|I_n(Y, \beta)|$. The open subscheme of $I_n(Y, \beta)$ defined by this open subset is $J_n(Y, C)$.

Definition 2.3 As $J_n(Y, C)$ is open in $I_n(Y, \beta)$, it has an induced (symmetric) obstruction theory and hence a virtual fundamental class of degree zero. Since $J_n(Y, C)$ is closed in $I_n(Y, \beta)$ it is projective, and so we can consider the degree of the virtual fundamental class

$$N_n(Y, C) = \deg [J_n(Y, C)]^{\text{vir}},$$

and call it the contribution of $C$ to the Donaldson-Thomas invariant $N_n(Y, \beta)$.

2.2 The closed subset $\bar{J}_n(Y, C)$

Definition 2.4 Let $C = \sum_i d_i C_i$ be as above and denote by supp$C$ the reduced closed subscheme of $Y$ underlying $C$. Let

$$\bar{J}_n(Y, C) \subset J_n(Y, C) \subset I_n(Y, \beta)$$

be the closed subset consisting of subschemes $Z \subset Y$ whose underlying closed subset $Z^{\text{red}} \subset Y$ is contained in supp$C$. 

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Remark 2.5 To see that $\tilde{J}_n(Y, C)$ is closed in $I_n(Y, \beta)$, let $W_m \subset Y$ be the $m$-th infinitesimal neighborhood of $\text{supp} C \subset Y$. For fixed numerical invariants $n$ and $\beta$ there exists a sufficiently large $m$, such that for any subscheme $Z \subset Y$, with invariants $n$ and $\beta$, and such that $Z^{\text{red}} \subset \text{supp} C$, we have $Z \subset W_m$. For such an $m$, consider the Hilbert scheme $I_n(W_m, \beta)$, which is a closed subscheme of $I_n(Y, \beta)$, as $W_m$ is a closed subscheme of $Y$. The underlying closed subset of $I_n(Y, \beta)$ is equal to $\tilde{J}_n(Y, C)$.

Remark 2.6 Informally speaking, $J_n(Y, C)$ parameterizes subschemes whose one dimensional components are confined to $C$, but may have embedded points anywhere in $Y$, whereas $\tilde{J}_n(Y, C)$ parameterizes subschemes where both the one dimensional components and the embedded points are supported on $C$.

2.3 The open Calabi Yau $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

We consider the open Calabi-Yau $N$, which is the total space of the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$. We denote by $C_0 \subset N$ the zero section. We consider the Hilbert scheme $I_n(N, [dC_0])$.

Let $\overline{N}$ denote $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$ and let $D_\infty = \overline{N} \setminus N$. Since $3D_\infty$ is an anti-canonical divisor of $\overline{N}$, the corresponding section defines a trivialization of $K_N$. $\overline{N}$ is naturally a toric variety, $D_\infty$ is an invariant divisor, and we let $T_0$ be the subtorus whose elements act trivially on $K_N$. Then $T_0$ induces a $T_0$-equivariant symmetric obstruction theory on $I_n(N, [dC_0])$, by Proposition 2.4 of [BF05]. Moreover, the $T_0$ fixed points in $I_n(N, [dC_0])$ are isolated points whose Zariski tangent spaces have no trivial factors as $T_0$ representations (the proof of Lemma 4.1, Part (a) and (b) in [BF05] is easily adapted to prove this).

As in [MNOP], the $T_0$ fixed points in $I_n(N, [dC_0])$ correspond to subschemes which are given by monomial ideals on the restriction to the two affine charts of $N$. The number of such fixed points is given by $p(n, d)$ described below.

Let $p(n, d)$ be the number of triples $(\pi_0, \lambda, \pi_\infty)$, where $\pi_0$ and $\pi_\infty$ are 3-dimensional partitions and $\lambda$ a 2-dimensional partition. The 3-dimensional partitions each have one infinite leg with asymptotics $\lambda$, and no other infinite legs. Moreover, $d = |\lambda|$ and $n$ is given by ([MNOP] Lemma 5)

$$n = |\pi_0| + |\pi_\infty| + \sum_{(i,j) \in \lambda} (i + j + 1),$$

where the size of a three dimensional partition with an infinite leg of shape $\lambda$ along the $z$ axis is defined by

$$|\pi| = \# \{(i, j, k) \in \mathbb{Z}^3_{\geq 0} : (i, j, k) \in \pi, (i, j) \not\in \lambda\}.$$

Proposition 2.7 We have

$$\tilde{\chi}(I_n(N, [dC_0])) = (-1)^{n-d}p(n, d).$$

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Proof. By Corollary 3.5 of [BF05], we have
\[ \chi(I_n(N, [dC_0])) = \sum_p (-1)^{\dim T_p}, \]
where the sum is over all \( T_0 \)-fixed points on \( I_n(N, [dC_0]) \) and \( T_p \) is the Zariski tangent space of \( I_n(N, [dC_0]) \) at \( p \). The parity of \( \dim T_p \) can be easily deduced from Theorem 2 of [MNOP] (just as in the proof of Lemma 4.1 (c) in [BF05]). The result is \( n - d \). So all we have to notice is that \( p(n, d) \) is the number of fixed points of \( T_0 \) on \( I_n(N, [dC_0]) \). □

Corollary 2.8 We have
\[ \chi(J_n(N, dC_0), I_n(N, [dC_0])) = (-1)^{n-d} p(n, d). \]
Proof. We just have to notice that all \( T_0 \)-fixed points are contained in \( J_n(N, dC_0) \). □

2.4 The box counting function \( p(n, d) \)
Counting three dimensional partitions with given asymptotics has been shown by Okounkov, Reshetikhin, and Vafa [ORV] to be equivalent to the topological vertex formalism which occurs in Gromov-Witten theory. They give general formulas for the associated generating functions in terms of \( q \) values of Schur functions which we will use to prove the following Lemma.

Lemma 2.9 The generating function for \( p(n, d) \) is given by
\[ \sum_{n=0}^{\infty} p(n, d) q^n = M(q)^2 P_d(q), \]
where the power series \( P_d(q) \) and \( M(q) \) are defined in Equations (1) and (2).

Proof. The generating function for the number of 3-dimensional partitions with one infinite leg of shape \( \lambda \) is given by equation 3.21 in [ORV]:
\[ \sum_{\text{3d partitions } \pi \text{ asymptotic to } \lambda} q^{\vert \pi \vert} = M(q)q^{-(\lambda)} \sum_{\lambda'} s_{\lambda'}(q^{1/2}, q^{3/2}, q^{5/2}, \ldots) \]
where \( \lambda' \) is the transpose partition, \( (\lambda') = \sum_i (\lambda_i) \), \( |\lambda| = \sum_i \lambda_i \), and
\[ s_{\lambda'}(q^{1/2}, q^{3/2}, q^{5/2}, \ldots) \]
is the Schur function associated to \( \lambda' \) evaluated at \( x_i = q^{(2i-1)/2} \). Using the homogeneity of Schur functions and writing
\[ s_{\lambda'}(q) = s_{\lambda'}(1, q, q^2, \ldots) \]
we can rewrite the right hand side of the above equation as
\[ M(q)q^{-(\lambda)} s_{\lambda'}(q). \]
Observing that
\[ \sum_{(i,j) \in \lambda} (i+j+1) = |\lambda| + \binom{\lambda}{2} + \binom{\lambda^t}{2}, \]
we get
\[ \sum_{n,d=0}^{\infty} p(n,d)q^n v^d = M(q)^2 \sum_{\lambda} s_{\lambda^t}(q)^2 q^{\lambda + \binom{\lambda^t}{2} - \binom{\lambda}{2}} v^{\lambda}. \]

The hook polynomial formula for \( s_{\lambda^t}(q) \) (I.3 ex 2 pg 45,[Mac95]) is
\[ s_{\lambda^t}(q) = q^{\binom{\lambda}{2}} \prod_{x \in \lambda^t} (1 - q^{h(x)})^{-1} \] (3)
from which one easily sees that
\[ s_{\lambda^t}(q) = q^{\binom{\lambda}{2} - \binom{\lambda^t}{2}} s_{\lambda}(q). \]

Therefore
\[ \sum_{n,d=0}^{\infty} p(n,d)q^n v^d = M(q)^2 \sum_{\lambda} s_{\lambda}(q) s_{\lambda^t}(q)^2 q^{\lambda + \binom{\lambda^t}{2} - \binom{\lambda}{2}} v^{\lambda} \]
\[ = M(q)^2 \sum_{\lambda} s_{\lambda}(q, q^2, q^3, \ldots) s_{\lambda^t}(v, vq, vq^2, \ldots) \]
\[ = M(q)^2 \prod_{i,j=1}^{\infty} (1 + q^{i+j-1} v) \]
where the last equality comes from the orthogonality of Schur functions (I.4 equation (4.3)' of [Mac95]). By rearranging this last sum and taking the \( v^d \) term, the lemma is proved. □

Remark 2.10 From the proof of the lemma we see that
\[ P_d(q) = q^d \sum_{\lambda \vdash d} s_{\lambda}(q) s_{\lambda^t}(q). \]
From Equation (3), it is immediate that \( P_d(q) \) is a rational function in \( q \). Moreover, using the formula for total hooklength (pg 11, I.1 ex 2, [Mac95]), it is easy to check that \( P_d(q) \) is invariant under \( q \mapsto 1/q \).

2.5 General \( Y \)

Lemma 2.11 Let \( C \) be a super-rigid rational curve on the Calabi-Yau threefold \( Y \). Then
\[ \chi(\bar{J}_n(Y, dC), J_n(Y, dC)) = (-1)^{n-d} p(n, d), \]
for all \( n, d \).
Proof. First of all, by Theorem 3.2 of [La81], an analytic neighborhood of $C$ in $Y$ is isomorphic to an analytic neighborhood of $C_0$ in $N$. Therefore, by the analytic theory of Hilbert schemes (or Douady spaces), see [Do66], we obtain an analytic isomorphism of $J_n(Y, dC)$ with $J_n(N, dC_0)$ which extends to an isomorphism of a tubular neighborhood of $J_n(Y, dC)$ in $I_n(Y, [dC])$ with a tubular neighborhood of $J_n(N, dC_0)$ in $I_n(N, [dC_0])$.

The formula for $\nu_X(P)$ in terms of a linking number, Proposition 4.22 of [Be05], shows that $\nu_X(P)$ is an invariant of the underlying analytic structure of a scheme $X$. Thus, we have

$$\tilde{\chi}(J_n(Y, dC), I_n(Y, [dC])) = \tilde{\chi}(J_n(N, dC_0), I_n(N, [dC_0])).$$

Finally, apply Corollary 2.8. □

Lemma 2.12 Let $f : X \to Y$ be an étale morphism of separated schemes of finite type over $\mathbb{C}$. Let $Z \subset X$ be a constructible subset. Assume that the restriction of $f$ to the closed points of $Z$, $f : Z(\mathbb{C}) \to Y(\mathbb{C})$, is injective. Then we have

$$\tilde{\chi}(f(X), Y) = \tilde{\chi}(Z, X).$$

We remark that by Chevalley’s theorem (EGA IV, Cor. 1.8.5), $f(Z)$ is constructible, so that $\tilde{\chi}(f(X), Y)$ is defined.

Proof. Without loss of generality, $Z \subset X$ is a closed subscheme and so $Z \to Y$ is unramified.

We claim that there exists a decomposition $Y = Y_1 \sqcup \ldots \sqcup Y_n$ into locally closed subsets, such that, putting the reduced structure on $Y_i$, the induced morphism $Z_i = Z \times_Y Y_i = Y_i$ is either an isomorphism, or $Z_i$ is empty.

In fact, by generic flatness (EGA IV, Cor. 6.9.3), we may assume without loss of generality that $Z \to Y$ is flat, hence étale. By Zariski’s Main Theorem (EGA IV, Cor. 18.12.13), we may assume that $Z \to Y$ is finite, hence finite étale. Then, by our injectivity assumption, the degree of $Z \to Y$ is 1 and so $Z \to Y$ is an isomorphism.

Once we have this decomposition of $Y$, the lemma follows from additivity of the Euler characteristic over such decompositions and the étale invariance of the canonical constructible function $\nu$. □

Now we consider the case of a curve with several components. Let

$$C_d = \sum_{i=1}^s d_i C_i$$

be an effective cycle, where the $C_i$ are pairwise disjoint super-rigid rational curves in $Y$. We assume $d_i > 0$, for all $i = 1, \ldots, s$.

For an $(s + 1)$-tuple of non-negative integers $\vec{m} = (m_0, m_1, \ldots, m_s)$, we let $|\vec{m}| = \sum_{i=0}^s m_i$. Consider, for $|\vec{m}| = n$ the open subscheme

$$U_{\vec{m}} \subset \text{Hilb}^{m_0}(Y) \times \prod_{i=1}^s J_{m_i}(Y, d_i C_i),$$

consisting of subschemes $(Z_0, (Z_i))$ with pairwise disjoint support.
Lemma 2.13  Mapping \((Z_0, (Z_i))\) to \(Z_0 \cup \bigcup_i Z_i\) defines an étale morphism

\[ f : U_{\vec{m}} \rightarrow J_n(Y, C_{\vec{d}}) \]

Proof. This is straightforward. See also Lemma 4.4 in [BF97]. \(\square\)

Let us write \(\check{Y}\) for \(Y \setminus \text{supp} C\) and remark that

\[ Z_{\vec{m}} = \text{Hilb}^{m_0}(\check{Y}) \times \prod_{i > 0} \tilde{J}_{m_i}(Y, d_i C_i) \]

is contained in \(U_{\vec{m}}\). Moreover, the restriction \(f : Z_{\vec{m}} \rightarrow J_n(Y, C_{\vec{d}})\) is injective on closed points. Finally, every closed point of \(J_n(Y, C_{\vec{d}})\) is contained in \(f(Z_{\vec{m}})\), for a unique \(\vec{m}\), such that \(|\vec{m}| = n\).

We will apply Lemma 2.12 to the diagram

Thus, we may calculate as follows:

\[
\bar{\chi}(J_n(Y, C_{\vec{d}})) = \sum_{|\vec{m}|=n} \bar{\chi}(f(Z_{\vec{m}}), J_n(Y, C_{\vec{d}}))
\]

\[
= \sum_{|\vec{m}|=n} \bar{\chi}(Z_{\vec{m}}, U_{\vec{m}})
\]

\[
= \sum_{|\vec{m}|=n} \bar{\chi}\left(\text{Hilb}^{m_0}(\check{Y}) \times \prod_{i > 0} \tilde{J}_{m_i}(Y, d_i C_i), \text{Hilb}^{m_0}(\check{Y}) \times \prod_{i > 0} J_{m_i}(Y, d_i C_i)\right)
\]

\[
= \sum_{|\vec{m}|=n} \bar{\chi}\left(\text{Hilb}^{m_0}(\check{Y}), \text{Hilb}^{m_0}(\check{Y}) \right) \prod_{i > 0} \bar{\chi}\left(\tilde{J}_{m_i}(Y, d_i C_i), J_{m_i}(Y, d_i C_i)\right)
\]

\[
= \sum_{|\vec{m}|=n} \bar{\chi}\left(\text{Hilb}^{m_0}(\check{Y})\right) \prod_{i > 0} (-1)^{m_i - d_i} p(m_i, d_i).
\]

Now we perform the summation:

\[
\sum_{n=0}^{\infty} \bar{\chi}(J_n(Y, C_{\vec{d}})) q^n
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{|\vec{m}|=n} \bar{\chi}\left(\text{Hilb}^{m_0}(\check{Y})\right) \prod_{i=1}^{s} (-1)^{m_i - d_i} p(m_i, d_i) \right) q^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{|\vec{m}|=n} \bar{\chi}(\text{Hilb}^{m_0}(\check{Y})) q^{m_0} \prod_{i=1}^{s} (-1)^{d_i} p(m_i, d_i)(-q)^{m_i}
\]

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\[
= \left( \sum_{\bf m} \chi(\text{Hilb}^{m_0}(\tilde{Y}))q^{m_0} \right) \sum_{m_i=0}^{s} \prod_{i=1}^{s} (-1)^{d_i} p(m_i,d_i)(-q)^{m_i}
\]
\[
= M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} \sum_{m_i=0}^{\infty} p(m_i,d_i)(-q)^{m_i}
\]
\[
= M(-q)^{\chi(Y)} \prod_{i=1}^{s} M(-q)^{2d_i} P_{d_i}(-q)
\]
\[
= M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q)
\]
\[
= M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q)
\]

By the main result of [Be05], Theorem 4.18, we have
\[
N_n(Y,C_{\tilde{d}}) = \chi(J_n(Y,C_{\tilde{d}})).
\]
This finishes the proof of:

**Theorem 2.14** The partition function for the contribution of $C_{\tilde{d}}$ to the Donaldson-Thomas invariants of $Y$ is given by
\[
Z(Y,C_{\tilde{d}}) = \sum_{n=0}^{\infty} N_n(Y,C_{\tilde{d}})q^n = M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q).
\]

**Corollary 2.15** Define the reduced partition function
\[
Z'(Y,C_{\tilde{d}}) = \frac{Z(Y,C_{\tilde{d}})}{Z(Y,0)}.
\]
Then we have
\[
Z'(Y,C_{\tilde{d}}) = \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q),
\]
a rational function in $q$, invariant under $q \mapsto 1/q$.

**Proof.** Behrend and Fantechi prove [BF05] that
\[
Z(Y,0) = \sum_{n=0}^{\infty} \chi(\text{Hilb}^{n}(Y))q^n = M(-q)^{\chi(Y)};
\]
the formula for $Z'(Y,C_{\tilde{d}})$ then follows immediately from Theorem 2.14. For the proof that $Z'(Y,C_{\tilde{d}})$ is a rational function invariant under $q \mapsto 1/q$, see Remark 2.10. □
3 The super-rigid GW/DT correspondence.

3.1 The usual GW/DT correspondence

The Gromov-Witten/Donaldson-Thomas correspondence of [MNOP] can be formulated as follows.

Let $Y$ be a Calabi-Yau threefold and let

$$Z_{DT}(Y, \beta) = \sum_{n \in \mathbb{Z}} N_n(Y, \beta) q^n$$

be the partition function for the degree $\beta$ Donaldson-Thomas invariants.

Let

$$Z'_{DT}(Y, \beta) = \frac{Z_{DT}(Y, \beta)}{Z_{DT}(Y, 0)}$$

be the reduced partition function.

In Gromov-Witten theory, the reduced partition function for the degree $\beta$ Gromov-Witten invariants, $Z'_{GW}(Y, \beta)$, is given by the coefficients of the exponential of the $\beta \neq 0$ part of the potential function:

$$1 + \sum_{\beta \neq 0} Z'_{GW}(Y, \beta) v^\beta = \exp \left( \sum_{\beta \neq 0} N^G_W(Y, \beta) v^{2g-2} \right) .$$  \hspace{1cm} (4)

Here

$$N^G_W(Y, \beta) = \deg([\overline{M}_g(Y, \beta)])^{\text{vir}}$$

is the genus $g$, degree $\beta$ Gromov-Witten invariant of $Y$.

The conjectural GW/DT correspondence states that

(i) The degree 0 partition function in Donaldson-Thomas theory is given by

$$Z_{DT}(Y, 0) = M(-q)^{\chi(Y)},$$

(ii) $Z'_{DT}(Y, \beta)$ is a rational function in $q$, invariant under $q \mapsto 1/q$, and

(iii) the equality

$$Z'_{GW}(Y, \beta) = Z'_{DT}(Y, \beta)$$

holds under the change of variables $q = -e^{iu}$.

Part (i) is proved for all $Y$ in [BF05].

3.2 The super-rigid GW/DT correspondence

In an entirely parallel manner, we can formulate the GW/DT correspondence for $N_n(Y, C_{\vec{d}})$, the contribution from a collection of super-rigid rational curves $C_t = \sum t_i C_i$.

Just as in Donaldson-Thomas theory, there is an open component of the moduli space of stable maps

$$\overline{M}_g(Y, C_{\vec{d}}) \subset \overline{M}_g(Y, \beta)$$
parameterizing maps whose image lies in the support of $C_{d}$. There are corresponding invariants given by the degree of the virtual class:

$$N_{g}^{GW}(Y, C_{d}) = \text{deg}[\overline{M}_{g}(Y, C_{d})]^{\text{vir}}$$

We define $Z'_{GW}(Y, C_{d})$ by replacing $N_{g}^{GW}(Y, \beta)$ on the right side of formula (4) by $N_{g}^{GW}(Y, C_{d})$.

Then we can formulate our results as follows.

**Theorem 3.1** The GW/DT correspondence holds for the contributions from super-rigid rational curves. Namely, let $C_{d} = d_{1}C_{1} + \cdots + d_{s}C_{s}$ be a cycle supported on pairwise disjoint super-rigid rational curves $C_{i}$ in a Calabi-Yau threefold $Y$, and let $Z'_{DT}(Y, C_{d})$ and $Z'_{GW}(Y, C_{d})$ be defined as above. Then

1. $Z'_{DT}(Y, C_{d})$ is a rational function of $q$, invariant under $q \mapsto 1/q$, and

2. the equality

$$Z'_{DT}(Y, C_{d}) = Z'_{GW}(Y, C_{d})$$

holds under the change of variables $q = -e^{iu}$.

**Proof.** For (ii), see Corollary 2.15. To prove (iii), we reproduce a calculation well known to the experts (e.g. [Ka06]).

By the famous multiple cover formula of Faber-Pandharipande [FP00] (see also [Pa99]),

$$N_{g}^{GW}(Y, C_{d}) = \sum_{i=1}^{s} c(g, d_{i}),$$

where $c(g, d)$ is given by

$$\frac{\sum_{g\geq 0} c(g, d)u^{2g-2}}{d} = \frac{1}{d} \left( \frac{\sin \left( \frac{du}{2} \right)}{u} \right)^{-2}.$$

We compute $Z'_{GW}(Y, C_{d})$ and make the substitution $q = -e^{iu}$:

$$1 + \sum_{(d_{1}, \ldots, d_{s}) \neq 0} Z'_{GW}(Y, C_{d})v_{1}^{d_{1}} \cdots v_{s}^{d_{s}}$$

$$= \exp \left( \sum_{j=1}^{s} \sum_{d_{j} > 1} c(g, d_{j})u^{2g-2} v_{j}^{d_{j}} \right)$$

$$= \prod_{j=1}^{s} \exp \left( \sum_{d_{j} > 1} \frac{v_{j}^{d_{j}}}{d_{j}} \left( -2 \sin \frac{d_{j}u}{2} \right)^{-2} \right)$$

$$= \prod_{j=1}^{s} \exp \left( \sum_{d_{j} > 1} \frac{v_{j}^{d_{j}}}{d_{j}} \left(-1 - e^{id_{j}u} \right)^{-2} \right)$$

$$= \prod_{j=1}^{s} \exp \left( \sum_{d_{j} > 1} \sum_{m_{j} > 1} \frac{-m_{j}}{d_{j}} e^{im_{j}u} v_{j}^{d_{j}} \right)$$

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\[
Y_j = 1 \exp \left( \sum_{m_j = 1}^{\infty} m_j \log \left( 1 - v_j e^{im_j u} \right) \right)
\]

\[
= \prod_{j=1}^{s} \prod_{m_j = 1}^{\infty} (1 - (-q)^{m_j} v_j)^{m_j}
\]

\[
= \prod_{j=1}^{s} \sum_{d_j = 0}^{\infty} P_{d_j} (-q)(-v_j)^{d_j}
\]

\[
= \sum_{\{d_1, \ldots, d_s\}} \prod_{j=1}^{s} (-1)^{d_j} P_{d_j} (-q)v_j^{d_j}.
\]

Therefore,

\[
Z_{GW}'(Y, C_d) = \prod_{j=1}^{s} (-1)^{d_j} P_{d_j} (-q)
\]

and so by comparing with Corollary 2.15 the theorem is proved. \(\square\)

The following corollary is immediate.

**Corollary 3.2** Let \(Y\) be a Calabi-Yau threefold and let \(\beta \in H_2(Y, \mathbb{Z})\) be a curve class such that all cycle representatives of \(\beta\) are supported on a collection of pairwise disjoint, super-rigid rational curves. Then the degree \(\beta\) \(GW/DT\) correspondence holds:

\[
Z_{DT}'(Y, \beta) = Z_{GW}'(Y, \beta).
\]

For example, we have:

**Corollary 3.3** Let \(Y \subset \mathbb{P}^4\) be a quintic threefold, and let \(L\) be the class of the line. Then for \(\beta\) equal to \(L\) or \(2L\), the \(GW/DT\) correspondence holds.

**Proof.** By deformation invariance of both Donaldson-Thomas and Gromov-Witten invariants, it suffices to let \(Y\) be a generic quintic threefold. It is well known that there are exactly 2875 pairwise disjoint lines on \(Y\) and they are all super-rigid. The conics on \(Y\) are all planar and hence rational, and it is known that there are exactly 609250 pairwise disjoint conics and they are super-rigid as well. For these facts and more, see [Ka86]. \(\square\)

Note that we cannot prove the \(GW/DT\) conjecture by this method for the quintic in degree three (and higher) due to the presence of elliptic curves in degree three.

Explicit formulas for the reduced Donaldson-Thomas partition function of a generic quintic threefold \(Y\) in degrees one and two are given below:

\[
Z_{DT}'(Y, L) = 2875 \frac{q}{(1 - q)^2}
\]

\[
Z_{DT}'(Y, 2L) = 609250 \frac{q}{(1 - q)^2} \cdot 2875 \cdot \frac{-2q^3}{(1 + q)^3(1 - q)^2}
\]

\[
= -3503187500 \frac{1}{(q - q^{-1})^4}
\]
References


