

Galois theory - symmetries of equations/numbers

Ex.

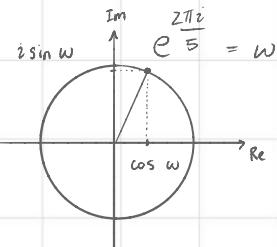
$\sin 72^\circ$ is an algebraic num. bc. it satisfies equation

$$x^4 - \frac{5}{4}x^2 + \frac{5}{16} = 0$$

$\in \mathbb{Q}[x]$

$$\text{since } 72^\circ = \frac{360^\circ}{5} = \frac{2\pi}{5} \quad \text{let } w = e^{\frac{2\pi i}{5}}$$

$$1 = w^5 = (a+ib)^5$$



$$\text{so } \sin 72^\circ = \text{Im } w = b$$

$$= a^5 + 5a^4ib + 10a^3i^2b^2 + 10a^2i^3b^3 + 5ai^4b^4 + i^5b^5$$

$$= a^5 + 5ia^4b - 10a^3b^2 - 10ia^2b^3 + 5ab^4 + ib^5$$

$$= (a^5 - 10a^3b^2 + 5ab^4) + i(5a^4b - 10a^2b^3 + b^5)$$

$$\Rightarrow 0 = 5a^4b - 10a^2b^3 + b^5$$

for $b \neq 0$

$$\Rightarrow 0 = 5a^4 - 10a^2b^2 + b^4$$

$$a^2 = 1 - b^2$$

$$= 5(1 - b^2) - 10(1 - b^2)b^2 + b^4$$

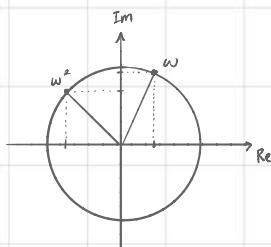
$$= 16b^4 - 20b^2 + 5 \quad (*)$$

$$\Rightarrow 0 = b^4 - \frac{5}{4}b^2 + \frac{5}{16} \quad \text{in fact this is irr. ie. min. deg. poly. with } b \text{ as root.}$$

algebraic formula for $b = \sin 72^\circ$:

$$b^2 = \frac{5}{8} \pm \frac{1}{2}\sqrt{\frac{25}{16} - \frac{20}{16}} = \frac{5}{8} \pm \frac{\sqrt{5}}{8} \quad x = \pm \sqrt{\frac{5}{8} \pm \frac{\sqrt{5}}{8}} \quad 4 \text{ roots ie. m. parts of the 4 non-trivial}$$

roots of $x^5 = 1$
the trivial solution
 $x = 1$ has $b = 0$ which we eliminated.



by inspection $\sin w > \sin(w^2)$ so $\sin w = \sqrt{\frac{5+\sqrt{5}}{8}}$

evidently by a similar argument, \sin or $\cos \frac{2\pi}{n}$ for $n \in \mathbb{N}$ is algebraic.

The symmetry gr. \tilde{G} of $(*)$ is cyclic of order 4 and acts on roots of $(*)$.

Better way to write these numbers:

$$\zeta = e^{\frac{2\pi i}{20}} \quad \text{so} \quad \zeta^{20} = 1$$

$$\zeta^4 = w \quad \zeta^5 = i \quad \zeta^{10} = -1$$

$$w = a + ib \quad \bar{w} = a - ib \quad w - \bar{w} = 2ib$$

$$b = \frac{w - \bar{w}}{2i} = \frac{\zeta^4 - \zeta^{-4}}{2\zeta^5} = \frac{1}{2} (\zeta^{-1} - \zeta^{-9}) = \frac{1}{2} (\zeta^{-1} + \zeta^{10}\zeta^{-9}) = \frac{1}{2} (\zeta + \zeta^{-1})$$

$\sin 72^\circ$

$$\text{similarly } \sin 144^\circ = \frac{1}{2} (\zeta^3 + \zeta^{-3})$$

$$\sin 216^\circ = \frac{1}{2} (\zeta^7 + \zeta^{-7})$$

$$\sin 288^\circ = \frac{1}{2} (\zeta^9 + \zeta^{-9})$$

$$\tilde{G} = (\mathbb{Z}/20\mathbb{Z})^\times \quad (\text{order } 8 = \varphi(20))$$

$$= \{\bar{1}, \bar{3}, \bar{7}, \bar{9}, \bar{11}, \bar{13}, \bar{17}, \bar{19}\}$$

acts on the four roots of $(*)$ by permuting the ms. st.:

1) respects algebraic operations $+, -, \times, \div$; and scalar mult. by \mathbb{Q}

2) $g(\zeta) = \zeta^g$ well-defined:

$$\text{ex. } \bar{3}(\zeta) = \zeta^3 = \zeta^{23} = \zeta^{43} = \dots = \zeta^{\bar{3}}$$

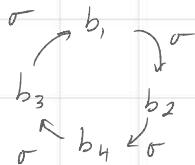
$$\text{ex. } \sigma = \bar{3}$$

$$\sigma(b_1) = \sigma\left(\frac{1}{2}(\zeta + \zeta^{-1})\right) = \frac{1}{2}(\sigma(\zeta) + \sigma(\zeta^{-1})) = \frac{1}{2}(\zeta^3 + \zeta^{-3}) = b_3$$

$\bar{-1}$ acts trivially

$$\text{thus } G \cong \tilde{G}/\langle \bar{-1} \rangle$$

cyclic of ord. 4.



Review of symmetry groups

Def. For $S \subset \mathbb{R}^n$, the symmetry gr. of S is
 $G = \{ \phi: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ isometry, } \phi(S) = S \}$ preserves distances

Lem. If $\varphi \in \text{Isom}(\mathbb{R}^n)$, $\exists! A \in O_n(\mathbb{R})$, $b \in \mathbb{R}^n$ st. $\varphi = Ax + b$

Def. $S, T \in \mathbb{R}^n$ have the same symmetry if $\exists \varphi \in \text{Isom}(\mathbb{R}^n)$ st.

$$\text{Sym}(S) = \text{Sym}(\varphi(T))$$

Rem. S, T having same sym. is stronger than $\text{Sym}(S) \cong \text{Sym}(T)$. E.g.

$$\begin{array}{ccc} S & & T \\ \text{sym} = \{\text{id}, \text{rot}\} & \cong & \text{sym} = \{\text{id}, \text{refl.}\} \end{array}$$

Prop. $S, T \in \mathbb{R}^n$ have same sym. iff $\text{Sym}(S), \text{Sym}(T) \stackrel{\text{sg}}{\subset} \text{Isom}(\mathbb{R}^n)$ are conj.

Def. (Semi-direct product)

Let N, H be gr., assume we have an action of H on N by gr. aut., i.e.

$$\begin{aligned} \varphi: H &\rightarrow \text{Aut}(N) \\ h &\mapsto \varphi(h) \end{aligned}$$

a gr. hom.

$$\begin{aligned} \varphi(h): N &\rightarrow N \\ n &\mapsto \varphi(h)(n) = {}^h n \end{aligned}$$

Then define an operation on $N \times H$:

$$(n, h) \cdot (n', h') = ({}^h n', h h')$$

This defines a gr. structure on $N \times H$, the semi-direct product of H, N , wrt. φ :

$$N \rtimes_{\varphi} H$$