Homework 6

Problem 1.
Let $K = \mathbb{C}(t)$ be a field extension of the complex numbers $\mathbb{C}$ generated by a transcendental element t. Determine the Galois groups over K of $x^n - t$, and $x^3 + t^2x - t^3$, and $x^4 - tx^2 + t^2$.

Problem 2.
Let L be a finite extension of the field K, of prime degree p. Suppose $L = K(\theta)$. Let $\theta = \theta_1, \ldots, \theta_p$ be the (distinct) conjugates of θ in a splitting field of its minimal polynomial containing L. Prove that if $\theta_2 \in L$, then L/K is Galois with cyclic Galois group or order p.

Problem 3.
Suppose that $f \in \mathbb{Q}[x]$ is of degree $n \geq 3$, with spitting field $K/\mathbb{Q}$. Suppose that $\mathrm{Gal}(K/\mathbb{Q}) = S_n$.

(a) Prove that f is irreducible.
(b) If $\alpha \in K$ is a root of f, then the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.
(c) Suppose $n \geq 4$. Prove that $\alpha^n \notin \mathbb{Q}$.

Problem 4.
Let $K/\mathbb{Q}$ be a finite extension. Prove that K contains only finitely many roots of unity.

Problem 5.
For which integers m does a primitive m-th rooth of unity have degree 2 over $\mathbb{Q}$?

Problem 6.
Let K be a field and $n \geq 1$ an odd integer. Prove that if K contains a primitive n-th root of unity, it also contains a primitive $2n$-th root of unity.

Problem 7.
Let G be a finite abelian group. Prove that there exists a Galois extension with Galois group G.

Problem 8.
Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. Let K be a maximal subfield of $\overline{\mathbb{Q}}$ not containing $\sqrt{2}$. Show that every finite extension of K is cyclic.