Homework 4

Problem 1.

(a) Find the minimal polynomial of the complex number $1 + i$ over $\mathbb{Q}$.

(b) Find the minimal polynomial of the real number $\cos 72^\circ$ over $\mathbb{Q}$.

Problem 2.

(a) Determine the degree $|\mathbb{Q}(2 + \sqrt{3}) : \mathbb{Q}|$.

(b) Determine the degree $|\mathbb{Q}\big(\sqrt{3 + 2\sqrt{2}}\big) : \mathbb{Q}|$.

Problem 3.

(a) Use the quadratic formula to prove that if $\operatorname{char} K \neq 2$, and L is a field extension of K such that $|L : K| = 2$, then there exists $a \in K^\times \setminus (K^\times)^2$, such that $L = K(\sqrt{a})$. (This notation is supposed to indicate that $(L, \sqrt{a})$ is a stem field for the irreducible polynomial $x^2 - a$ over K.)

(b) Give an example of a field extension $K \subset L$ of degree 2, such that there does not exist any element $a \in L \setminus K$, such that $a^2 \in K$.

Problem 4.

Use the triple angle formula for the cosine to prove that the angle θ may be trisected with straight-edge and compass if and only if the polynomial $4x^3 - 3x - \cos\theta$ has a root over $\mathbb{Q}(\cos\theta)$.

Problem 5.

Call a real number *superconstructible*, if it can be constructed with straight edge, compass, and an angle trisector. An angle trisector is a device that allows you to construct, for any $0 < \theta < \pi$, the number $\cos\frac{1}{3}\theta$ from $\cos\theta$.

Let $K \subset \mathbb{R}$ be a field extension of $\mathbb{Q}$, consisting of constructible numbers. Let $L = K(\alpha)$, where $4\alpha^3 - 3\alpha - \beta = 0$, for some $\beta \in K$, such that $-1 < \beta < 1$. Prove that the elements of L are superconstructible numbers.

Deduce that $\cos(2\pi/7)$ is superconstructible. So the regular heptagon can be constructed with straight edge compass and angle trisector.