Math 422/501, Winter 2019, Term 1

Homework 1

Problem 1.
Let B_n be the subgroup of upper triangular matrices in $GL_n(\mathbb{R})$, let T_n be the group of diagonal matrices in $GL_n(\mathbb{R})$, and let $U_n \subset B_n$ be the subgroup of matrices all of whose diagonal entries equal 1. Prove that $B_n = U_n \rtimes T_n$.

Problem 2.
Prove that every isometry of $\mathbb{R}^2$ is one of three types:
(a) a translation,
(b) a rotation,
(c) a glide reflection.
(Recall that a glide reflection is an isometry of the form $T \circ S$, where S is a reflection across a line L, and T is a translation in a direction parallel to L.)

Problem 3.
Find three objects in $\mathbb{R}^3$, all with different symmetry, but whose symmetry groups are isomorphic to D_8, the dihedral group with 8 elements.

Problem 4.
(a) Let $G \subset I_n$ be the symmetry group of a subset of $\mathbb{R}^n$. Let $T \subset G$ be the subgroup of translations. Prove that $T \subset G$ is a normal subgroup. The **point group** of G is defined to be the quotient $\overline{G} = G/T$. Construct an injective homomorphism $\overline{G} \to O_n$. Prove that conjugation induces an action of $\overline{G}$ on T, and that, via the embeddings $\overline{G} \subset O_n$ and $T \subset \mathbb{R}^n$, this conjugation action is given by matrix multiplication. (Careful: in general, G will not be the semi-direct product of T and $\overline{G}$.)
(b) Now assume that $n = 2$, and that there exist two linearly independent vectors $b_1, b_2 \in \mathbb{R}^2$, such that $T = \mathbb{Z}b_1 + \mathbb{Z}b_2$. Prove the **crystallographic restriction**, namely that every rotation in $\overline{G} \subset O_2$ has order 1, 2, 3, 4 or 6. Deduce that, in this case, the point group is one of C_1, C_2, C_3, C_4, C_6, D_2, D_4, D_6, D_8, D_{12}.
(c) Find the point groups of the following four patterns (extended infinitely in all directions):