Final Exam

Monday, December 18, 2017

No books, notes or calculators

Problem 1.
Define the following terms:
(a) polynomial solvable in radicals
(b) splitting field of a polynomial
(c) discriminant of a polynomial
(d) Galois group of a polynomial
(e) constructible number
(f) algebraic closure of a field

Problem 2.
Carefully state:
(a) fundamental theorem of Galois theory
(b) Galois' theorem characterizing polynomials solvable in radicals in terms of their Galois group
(c) the main theorem of Kummer theory, characterizing certain kinds of abelian extensions in terms of data associated to the ground field
(d) a result about the Galois group in terms of the discriminant.

Problem 3.
Give examples of the following phenomena (without proofs):
(a) a polynomial of degree 3 with cyclic Galois group over $\mathbb{Q}$
(b) a polynomial of degree 11 with cyclic Galois group over $\mathbb{C}(t)$
(c) a polynomial of degree 178 over $\mathbb{Q}$, which is irreducible
(d) a field extension whose Galois group is S_4
(e) an inseparable polynomial
(f) a field extension of degree 4 without any (proper) intermediate fields

Problem 4.
Find the degree of the splitting field of $x^6 - 4$ over $\mathbb{Q}$.

Problem 5.
Determine structure of the Galois group of $(x^4 + 3)(x^3 - 2)$ over $\mathbb{Q}(\sqrt{-3})$.

Problem 6.
Are the splitting fields of the polynomials $x^3 - 3$ and $x^3 - 2$ isomorphic over $\mathbb{Q}$? Justify your answer.

Problem 7.
Let L/K be a field extension, and $\alpha \in L$ an element. Define the trace of α, notation $\mathrm{tr}_{L/K}(\alpha)$, to be the trace of the K-linear operator $m_\alpha : L \to L$, defined by $m_\alpha(x) = \alpha x$, for $x \in L$. Note (do not prove) that $\mathrm{tr}_{L/K}$ is a K-linear map $L \to K$.

(a) Compute $\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(a + ib)$ and $\mathrm{tr}_{\mathbb{Q}(\sqrt[3]{3})/\mathbb{Q}}(a + b\sqrt[3]{3} + c\sqrt[3]{9})$.

(b) Suppose that $L = K(\alpha)$, and that the minimal polynomial of α over K is $\sum_{i=0}^{n} a_i x^i$. Prove that $\mathrm{tr}_{L/K}(\alpha) = -a_{n-1}$.

(c) Suppose that α is inseparable and $L = K(\alpha)$. Prove that $\mathrm{tr}_{L/K}\,\alpha = 0$.

(d) Suppose that α is separable, that $L = K(\alpha)$, and that M/K is a field extension in which the minimal polynomial of α splits completely. Prove that $\mathrm{tr}_{L/K}\,\alpha = \sum_{\sigma: L \to M} \sigma(\alpha)$.

Hint. The theorem of Cayley-Hamilton says that if V is a finite-dimensional vector space over the field K, and $\phi : V \to V$ is a K-linear endomorphism of V, then the morphism of K-algebras $K[x] \to \mathrm{End}_K(V)$ defined by $x \mapsto \phi$ annihilates the characteristic polynomial of ϕ.

Problem 8.
Let p be a prime number. Prove that the Galois group of the splitting field of $x^p - p \in \mathbb{Q}[x]$ is isomorphic to the semi-direct product

$$\mu_p \rtimes \mathrm{Aut}(\mu_p)\,,$$

where μ_p is a cyclic group of order p, and $\mathrm{Aut}(\mu_p)$ is the group of automorphisms of the group μ_p, which acts on μ_p in the canonical way.