Hints for Homework 5

3.3 Let \( a = 2 + \sqrt{2}, \) and \( \overline{a} = 2 - \sqrt{2}, b = 3 + \sqrt{3}, \overline{b} = 3 - \sqrt{3}. \) Finally, let \( \gamma = \sqrt{ab}. \)

Prove that \( \mathbb{Q}(a, b, \gamma) \) is a Galois extension of \( \mathbb{Q} \) by proving that the polynomial

\[
(x^2 - ab)(x^2 - a\overline{b})(x^2 - b\overline{a})(x^2 - \overline{a}b)
\]

has rational coefficients, and that it splits completely in \( \mathbb{Q}(a, b, \gamma). \) You do not need to compute the coefficients of this polynomial to do this. Rather, you use everything you know about the Galois extension \( \mathbb{Q} \subset \mathbb{Q}(a, b). \)

To prove that the Galois group of \( K(a, b, \gamma) \) is the quaternion group, rather than constructing an explicit isomorphism between the two groups, you could try using the fact that the quaternion group is the only group of order 8 with 3 different cyclic subgroups of order 4. This could be done by proving that the extension \( \mathbb{Q}(\sqrt{c}) \subset \mathbb{Q}(a, b, \gamma) \) is cyclic, for \( c = 2, 3, 6. \)