Homework 7

Problem 1.
Let $K = \mathbb{C}(t)$ be a field extension of the complex numbers $\mathbb{C}$ generated by a transcendental element $t$. Determine the Galois groups over $K$ of $x^n - 1$, and $x^3 + t^2 x - t^3$, and $x^4 - tx^2 + t^2$.

Problem 2.
Let $L$ be a finite extension of the field $K$, of prime degree $p$. Suppose $L = K(\theta)$. Let $\theta = \theta_1, \ldots, \theta_p$ be the conjugates of $\theta$ in a splitting field of its minimal polynomial containing $L$. If $\theta_2 \in L$, then $L/K$ is Galois with cyclic Galois group or order $p$.

Problem 3.
Suppose that $f \in \mathbb{Q}[x]$ is of degree $n$, with splitting field $K/\mathbb{Q}$. Suppose that $\text{Gal}(K/\mathbb{Q}) = S_n$.

(a) Prove that $f$ is irreducible.

(b) If $\alpha \in K$ is a root of $f$, then the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.

(c) Suppose $n \geq 4$. Prove that $\alpha^n \not\in \mathbb{Q}$.

Problem 4.
Let $K/\mathbb{Q}$ be a finite extension. Prove that $K$ contains only finitely many roots of unity.

Problem 5.
For which integers $m$ does a primitive $m$-th root of unity have degree 2 over $\mathbb{Q}$?

Problem 6.
Let $K$ be a field and $n \geq 1$ an odd integer. Prove that if $K$ contains a primitive $n$-th root of unity, it also contains a primitive $2n$-th root of unity.

Problem 7.
Let $G$ be a finite abelian group. Prove that there exists a Galois extension with Galois group $G$.

Problem 8.
Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. Let $K$ be a maximal subfield of $\overline{\mathbb{Q}}$ not containing $\sqrt{2}$. Show that every finite extension of $K$ is cyclic.