Midterm Exam I

October 3, 2008

No books. No notes. No calculators. No electronic devices of any kind.

Name ____________________ **Student Number** ____________________

Problem 1. (5 points)
The cone with equation $y^2 + z^2 = x^2$ and the plane with equation $x + z = 4$ intersect in a curve C. Find the curvature of C at the point $\langle 2, 0, 2\rangle$.

use $t = y$. Then $t^2 + z^2 = x^2$ and $x + z = 4$, which we solve for x and z to get $x = 2 + \frac{1}{8}t^2$ $\quad y = t \quad$ $z = 2 - \frac{1}{8}t^2$.

So $\vec{r}(t) = \langle 2 + \frac{1}{8}t^2, t, 2 - \frac{1}{8}t^2 \rangle$

$\vec{r}\,'(t) = \langle \frac{1}{4}t, 1, -\frac{1}{4}t \rangle$

$\vec{r}\,''(t) = \langle \frac{1}{4}, 0, -\frac{1}{4} \rangle$.

The point $\langle 2,0,2 \rangle$ corresponds to $t = 0$:

$\vec{r}\,'(0) = \langle 0, 1, 0 \rangle$

$\vec{r}\,''(0) = \langle \frac{1}{4}, 0, -\frac{1}{4} \rangle$.

So $$\kappa(0) = \frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3} = \frac{|\langle 0,1,0\rangle \times \langle \frac{1}{4}, 0, -\frac{1}{4}\rangle|}{|\langle 0,1,0\rangle|^3} = \frac{|\langle -\frac{1}{4}, 0, -\frac{1}{4}\rangle|}{1^3} = \frac{\sqrt{2}}{4}.$$

Problem 1. continued.

Remark 1. If you use the formula $\kappa(0) = \frac{|\vec{T}'(0)|}{|\vec{r}'(0)|}$

you have to be very careful:

$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ At this point you cannot substitute $|\vec{r}'(0)|$ in the denominator, because the formula for $\kappa(0)$ still requires you to differentiate $\vec{T}(t)$.

So you cannot use $\vec{T}(t) = \langle \frac{1}{4}t, 1, -\frac{1}{4}t \rangle$

instead you have to use:

$$\vec{T}(t) = \frac{1}{\sqrt{t^2/8+1}} \langle \tfrac{1}{4}t, 1, -\tfrac{1}{4}t \rangle.$$

(Only because, by coincidence, the derivative of $\frac{1}{\sqrt{t^2/8+1}}$ vanishes at $t=0$ do you still get the correct answer.)

Remark 2. If you use $x=t$, you get $\vec{r}(t) = \langle t, \sqrt{8t-16}, 4-t \rangle$, which requires $8t-16 \geq 0$ or $t \geq 2$. If you want to differentiate, it requires $t > 2$. (The domain of $\vec{r}'(t) = \langle 1, \frac{8}{\sqrt{8t-16}}, -1 \rangle$ is $t > 2$.)

So this parametrization is not suitable for computing $\kappa(2)$.

(It corresponds to speeding up to infinite speed as $t \searrow 2$.)

1	2	3	4	5	6	total/22

Problem 2. (6 points)
The spiral C in the plane is parametrized by the vector funtion

$$\vec{r}(t) = e^t\langle \cos t, \sin t\rangle$$

(a) Find the arclength of the part of C which is parametrized by the interval $(-\infty, 0]$.
(b) Reparametrize C using arc-length measured from $t = -\infty$.

$\vec{r}'(t) = e^t < \cos t, \sin t > + e^t < -\sin t, \cos t >$ by product rule.

$= e^t < \cos t - \sin t, \cos t + \sin t >$

$|\vec{r}'(t)| = e^t \, |< \cos t - \sin t, \cos t + \sin t >| = e^t \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2}$

$= \sqrt{2}\, e^t$.

(a) $L = \int_{-\infty}^{0} |\vec{r}'(u)|\, du = \int_{-\infty}^{0} \sqrt{2} e^u\, du = \sqrt{2} e^u \Big|_{-\infty}^{0} = \sqrt{2} - 0 = \sqrt{2}$.

this calculation is justified, because $\lim_{u \to -\infty} \sqrt{2} e^u = 0$.

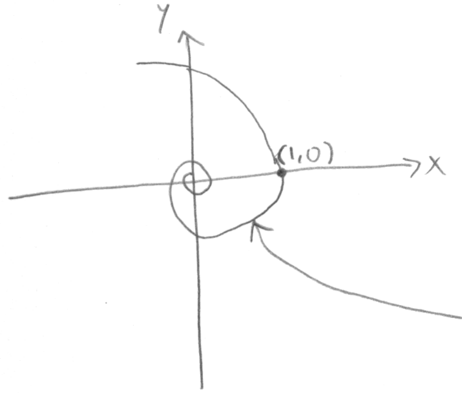
(b) $s = \int_{-\infty}^{t} |\vec{r}'(u)|\, du = \int_{-\infty}^{t} \sqrt{2} e^u\, du = \sqrt{2} e^u \Big|_{-\infty}^{t} = \sqrt{2} e^t - 0 = \sqrt{2} e^t$.

Solve $s = \sqrt{2} e^t$ for t: $e^t = \frac{\sqrt{2}}{2} s$ $t = \ln(\frac{\sqrt{2}}{2} s)$

So

$\vec{r}(s) = \frac{\sqrt{2}}{2} s < \cos \ln \frac{\sqrt{2}}{2} s, \sin \ln \frac{\sqrt{2}}{2} s >$ with domain $s > 0$.

Problem 2 cont'd.



Note that the spiral spirals towards the origin so quickly that the total arclength of this part is finite $(=\sqrt{2})$, even though the spiral spirals around the origin infinitely many times.

Note that the original parametrization had domain $t \in (-\infty, \infty) = \mathbb{R}$
the new parametrization has domain $s \in (0, \infty)$ or $s > 0$.

For (b): We know that $\text{speed} = |\vec{r}\,'(t)| = \dfrac{ds}{dt} = \sqrt{2}e^t$,

so $s = \sqrt{2}e^t + C$ (indefinite instead of definite (improper) integral.)

You still need to justify why the constant of integration is $C = 0$.

For example, set $t = 0$: $s(0) = \sqrt{2} + C$, but $s(0) = \sqrt{2}$ from (a),

so $C = 0$.

Problem 3. (3 points)
True or false? (Assumme that a curve C is parametrized by a twice continuously differentiable vector function $\vec{r}(t)$.)

(a) at a time t where $|\vec{v}(t)|$ reaches a maximum, we necessarely have $\vec{a}(t) \perp \vec{v}(t)$.
(b) at a time t where $|\vec{v}(t)|$ is not zero and $\vec{a}(t) \parallel \vec{v}(t)$, the curvature $\kappa(t)$ vanishes.
(c) at a time t where $|\vec{v}(t)|$ vanishes, we must have that $\vec{a}(t)$ is tangent to the curve.

All of these can be solved with the formula

$$\vec{a} = v' \vec{T} + \kappa v^2 \vec{N}.$$

(a) if v reaches a maximum, then $v' = 0$, so $\vec{a} \parallel \vec{N}$, so $\vec{a} \perp \vec{T}$, so $\vec{a} \perp \vec{v}$. TRUE.

note that this remains true, even if $\vec{N}$ is not defined, because $\kappa = 0$. (in this case we still have $\vec{a} = v'\vec{T}$, so $v' = 0$ implies $\vec{a} = \vec{0}$, and so $\vec{a} \perp \vec{v}$ because $\vec{0}$ is $\perp$ to every vector)

(note also, that if $\vec{T}$ is not defined because $v = 0$, then $v = 0$ being a maximum means that $v = 0$ everywhere, so that no movement is taking place, $\vec{a} = \vec{0}$ and $\vec{v} = \vec{0}$ in this case, still we have $\vec{a} \perp \vec{v}$.)

(b) $\vec{a} \parallel \vec{v}$ means $\vec{a} \parallel \vec{T}$ so $a_N = 0$, so $\kappa v^2 = 0$. But then $v^2 = 0$ implies $\kappa = 0$. TRUE.

[You may object that $\kappa = 0$ means that $\vec{N}$ is not defined, but that objection is not relevant. If both v and κ are not zero, then $\vec{T}$ and $\vec{N}$ and the above formula are defined, and the argument (b) shows that we get a contradiction, which (as $v \neq 0$) can only be resolved by $\kappa = 0$.]

(c) if $v = 0$, then $a_N = 0$ and so $\vec{a} \parallel \vec{T}$.

TRUE.

(If $\vec{a} \nparallel \vec{T}$ then $a_N \neq 0$ and so $\kappa \neq 0$ and $v \neq 0$.)

If you object, because $v = 0$ means the formulas for a_T and a_N don't work, here is a more detailed argument.

There has to be some parameter u, such that $\frac{d\vec{r}}{du} \neq \vec{0}$

We have

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{du}\frac{du}{dt}\right) = \frac{d^2\vec{r}}{du^2}\left(\frac{du}{dt}\right)^2 + \frac{d\vec{r}}{du}\frac{d^2u}{dt^2}$$

and $\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{du}\frac{du}{dt}$ so $v = 0$ implies $\vec{v} = \frac{d\vec{r}}{dt} = 0$, implies $\frac{du}{dt} = 0$ (as $\frac{d\vec{r}}{du} \neq \vec{0}$).

so $\vec{a} = \frac{d\vec{r}}{du}\frac{d^2u}{dt^2} \parallel \frac{d\vec{r}}{du}$ which is a tangent vector to C.

Important Remark: in (a) (b) (c) we are referring to one point in time, not all pts in time.

For example (b): if $\vec{a} \parallel \vec{v}$ always then $\kappa = 0$ always $\to$ line

but if we just have $\vec{a}(0) \parallel \vec{v}(0)$ we have $\kappa(0) = 0$,

but C does not have to be a line.

(c): if $\vec{v}(0) = \vec{0}$ it does not follow that $\vec{a}(0) = \vec{0}$.

Problem 4. (4 points)
The curve C is parametrized by the vector funtion

$$\vec{r}(t) = \langle t, e^{-t}, \cos t \rangle$$

Find an equation for the normal plane to C at the point corresponding to the parameter value $t = 0$.

$\vec{r}'(t) = \langle 1, -e^{-t}, -\sin t \rangle$

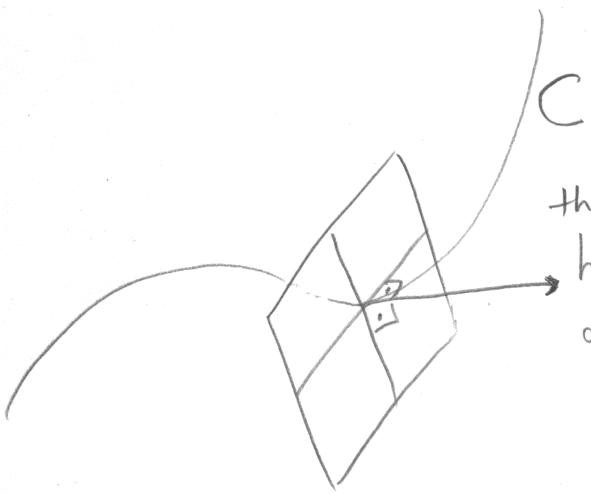
$\vec{r}'(0) = \langle 1, -1, 0 \rangle$ tangent to C, normal to normal plane,

So equation is

$$\langle 1, -1, 0 \rangle \cdot \left(\langle x, y, z \rangle - \langle 0, 1, 1 \rangle \right) = 0$$

$$\langle 1, -1, 0 \rangle \cdot \langle x, y-1, z-1 \rangle = 0$$

$$\boxed{x - y + 1 = 0}$$



the normal plane to C has a tangent vector to C as normal vector.

Problem 5. (4 points)

The derivative $\frac{d}{dt}|\vec{r}'(t)|$ is given by

(a) $|\vec{r}''(t)|$,

(b) $2\vec{r}'(t) \cdot \vec{r}''(t)$,

$\longrightarrow$ (c) $\dfrac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$,

(d) 0

(e) non of the above.

The derivative $\frac{d}{dt}\Big(\vec{r}(t) \times \vec{r}'(t)\Big)$ is equal to

(a) $\vec{r}'(t) \times \vec{r}'(t)$,

$\longrightarrow$ (b) $\vec{r}(t) \times \vec{r}''(t)$,

(c) $\vec{r}'(t) \times \vec{r}''(t) + \vec{r}(t) \times \vec{r}''(t)$

(d) $\vec{r}(t) \times \vec{r}'(t) + \vec{r}(t) \times \vec{r}''(t)$

(e) none of the above.

$$\frac{d}{dt}|\vec{r}'| = \frac{d}{dt}\sqrt{\vec{r}'\cdot\vec{r}'} = \frac{1}{2\sqrt{\vec{r}'\cdot\vec{r}'}}\,\frac{d}{dt}(\vec{r}'\cdot\vec{r}')$$

$$= \frac{1}{2|\vec{r}'|}\left(\vec{r}''\cdot\vec{r}' + \vec{r}'\cdot\vec{r}''\right) = \frac{2\,\vec{r}'\cdot\vec{r}''}{2|\vec{r}'|} = \frac{\vec{r}'\cdot\vec{r}''}{|\vec{r}'|} \quad (c).$$

$$\frac{d}{dt}(\vec{r}\times\vec{r}') = \underbrace{\vec{r}'\times\vec{r}'}_{\vec{0}} + \vec{r}\times\vec{r}'' = \vec{r}\times\vec{r}'' \quad (b).$$

b/c $\vec{r}' \parallel \vec{r}'$