Midterm Exam II

November 7, 2012

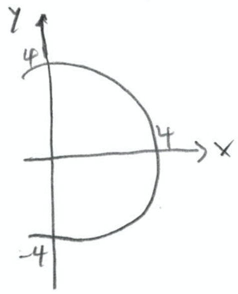
No books. No notes. No calculators. No electronic devices of any kind.

Name ____________________ **Student Number** ____________________

Problem 1. (5 points)
Evaluate the line integral

$$\int_C xy^4\, ds\,,$$

where C is the right half of the circle $x^2 + y^2 = 16$ in the plane.



use y as parameter: $-4 \le t \le 4$

$x = \sqrt{16 - t^2}$

$y = t$

$$ds = \left|\frac{d\vec{r}}{dt}\right| dt = \left|\left\langle \frac{-t}{\sqrt{16-t^2}}, 1\right\rangle\right| dt = \sqrt{\frac{t^2}{16-t^2} + 1}\; dt = \frac{4}{\sqrt{16-t^2}}\, dt$$

$$\int_C xy^4\, ds = \int_{-4}^{4} \sqrt{16-t^2}\; t^4\, \frac{4}{\sqrt{16-t^2}}\, dt = 4\int_{-4}^{4} t^4\, dt = \frac{4}{5} t^5 \Big|_{-4}^{4} = \frac{4}{5}\left(4^5 - (-4)^5\right)$$

$$= \frac{4}{5}\, 2\cdot 4^5 = \frac{2}{5} 4^6 = \frac{2^{13}}{5}$$

(sorry for the big number!)

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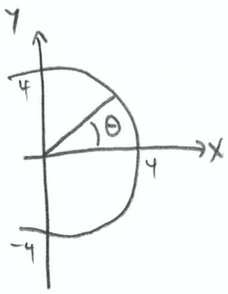


use θ as parameter: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$x = 4\cos\theta$

$y = 4\sin\theta$

$ds = \left|\frac{d\vec{r}}{d\theta}\right| d\theta \cdot |\langle -4\sin\theta, 4\cos\theta\rangle|\, d\theta = 4\,|\langle -\sin\theta, \cos\theta\rangle\, d\theta$

$= 4\, d\theta$

$$\int_C xy^4 ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos\theta\,(4\sin\theta)^4\, 4\, d\theta = 4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta)^4 \cos\theta\, d\theta$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$$= 4^6 \int_{-1}^{1} u^4\, du = 4^6\, \frac{u^5}{5}\Big|_{-1}^{1} = 4^6\left(\frac{1}{5} - \left(\frac{-1}{5}\right)\right) = \frac{2}{5}4^6 = \frac{2^{13}}{5}$$

1	2	3	4	5	total/25

Problem 2. (5 points)
Use the fundamental theorem to evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r},$$

where C is the parametrized curve

$$\vec{r}(t) = \langle t^2, t\pi, 2t+1 \rangle, \qquad 0 \le t \le 1,$$

and $\vec{F}$ is the vecctor field

$$\vec{F}(x,y,z) = \langle y\cos(xy), x\cos(xy) + z\sin(yz), y\sin(yz) \rangle.$$

Find the numerical value of the integral.

The fundamental theorem for line integrals only applies to conservative vector fields. So $\vec{F}$ should be conservative.

Check that $\vec{F}$ is conservative by finding a potential $f(x,y,z)$:

(1) $\frac{\partial f}{\partial x} = y\cos(xy)$

(2) $\frac{\partial f}{\partial y} = x\cos(xy) + z\sin(yz)$

(3) $\frac{\partial f}{\partial z} = y\sin(yz)$

integrate (1) with respect to x: $f = \sin(xy) + C(y,z)$

differentiate the result with respect to y and z: $\frac{\partial f}{\partial y} = x\cos(xy) + \frac{\partial C}{\partial y}$

$\frac{\partial f}{\partial z} = \frac{\partial C}{\partial z}$

(cont'd)

Overflow space I.

compare with (2) and (3) :

$$x\cos(xy) + z\sin(yz) = \frac{\partial f}{\partial y} = x\cos(xy) + \frac{\partial C}{\partial y}$$

$$y\sin(yz) = \frac{\partial f}{\partial z} = \frac{\partial C}{\partial z}$$

to get (4) $\frac{\partial C}{\partial y} = z\sin(yz)$

(5) $\frac{\partial C}{\partial z} = y\sin(yz)$

Now we have eliminated x, only variables left are y, z.

integrate (4) with respect to y: $C = -\cos(yz) + E(z)$

differentiate with respecto to z: $\frac{\partial C}{\partial z} = y\sin(yz) + \frac{dE}{dz}$

compare with (5) : $y\sin(yz) = y\sin(yz) + \frac{dE}{dz}$

so $\frac{dE}{dz} = 0$, E is constant.

So $C = -\cos(yz) + E$

and $f = \sin(xy) - \cos(yz) + E$.

We do not need all potentials, only one, so set $E = 0$.

$f = \sin(xy) - \cos(yz)$ is a potential for $\vec{F}$.

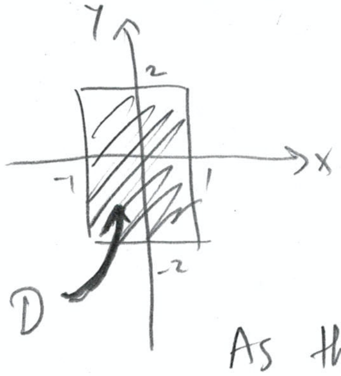
So $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f\Big|_{\vec{r}(0)}^{\vec{r}(1)} = \big(\sin(xy) - \cos(yz)\big)\Big|_{\langle 0,0,1\rangle}^{\langle 1,\pi,3\rangle}$

$= \sin(\pi) - \cos(3\pi) - \big(\sin(0) - \cos(0)\big) = 0 - (-1) - 0 + 1 = 2.$

Problem 3. (5 points)
Use Green's theorem to evaluate the line integral

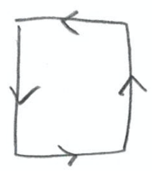
$$\oint_C \langle e^y, 2xe^y \rangle \cdot \langle dx, dy \rangle \,,$$

Where C is the rectangle with sides $x = \pm 1$, $y = \pm 2$.



$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \, dA$$

As the orientation of C is not specified, we assume it's the orientation as boundary of D, i.e. the counterclockwise orientation



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Then $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \iint_D \frac{\partial 2xe^y}{\partial x} - \frac{\partial e^y}{\partial y} \, dA$

$$= \iint_D (2e^y - e^y) \, dA = \iint_D e^y \, dA = \int_{x=-1}^{x=+1} \int_{y=-2}^{y=2} e^y \, dy \, dx = \int_{-2}^{2} e^y \, dy \int_{-1}^{1} dx$$

$$= 2 e^y \Big|_{-2}^{2} = 2\left(e^2 - e^{-2}\right)$$

Problem 4. (6 points)
Assume that the vector field $\vec{F}$ has continuous partial derivatives everywhere in the domain D, where D is $\mathbb{R}^2$ without the origin. True or false?

(a) If $\text{curl}(\vec{F}) = 0$ throughout D, then $\vec{F}$ is conservative,

(b) if $\text{curl}(\vec{F}) = 0$ throughout the first quadrant, then $\vec{F}$ is conservative in the first quadrant,

(c) If $\text{curl}(\vec{F}) = 2$ throughout D, then the line integral $\int_{C_1} \vec{F} \cdot d\vec{r} > \int_{C_2} \vec{F} \cdot d\vec{r}$, where C_1 and C_2 are semicircles from $(2, -1)$ to $(2, 1)$, and C_1 goes to the left, and C_2 to the right of the point $(2, 0)$.

(d) If $\text{curl}(\vec{F}) < 0$, throughout D, and $\vec{F}$ is the velocity field of a fluid flow, then the fluid will push any chain of boats constrained to move along a simple closed curve which does not enclose the origin, in a clockwise direction.

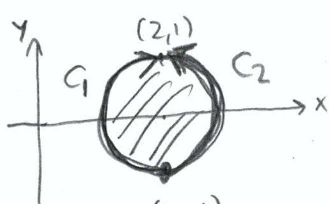
(e) if $\text{curl}(\vec{F}) = 3$ throughout D, then the integral $\oint \vec{F} \cdot d\vec{r}$, where C is a circle of radius 2 around the origin (oriented counterclockwise) is equal to 12π.

(f) if $\text{curl}(\vec{F}) = 3$ throughout D, then any paddle wheel inserted into the fluid in such a way that it avoids the origin, will rotate counterclockwise, with an angular speed of 6 radians per unit time.

(a) False $\quad \vec{F} = \dfrac{\langle -y, x\rangle}{x^2+y^2}$ is a counterexample: curl $\vec{F} = 0$ everywhere in D, but $\vec{F}$ is not conservative b/c $\oint_{\substack{\text{circle} \\ \text{around } 0}} \vec{F}\cdot d\vec{r} = 2\pi$.

(b) True. The first quadrant  is simply connected, so curl $\vec{F} = 0$ implies that $\vec{F}$ is conservative.

(c) False.

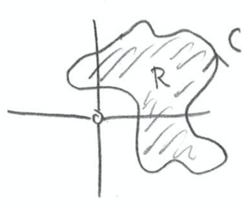


By Green's theorem,

$$\int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C_1} \vec{F}\cdot d\vec{r} = 2\iint_{\text{Disk}} dA = 2\pi$$

so $\int_{C_2} > \int_{C_1}$.

(d) True.



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\vec{F}\, dA < 0$$ So circulation is < 0

which means clockwise movement.

Overflow space II.

(e) False. For example, the vector field $\vec{F} = \frac{\langle -y, x\rangle}{x^2+y^2} + \frac{3}{2}\langle -y, x\rangle$

has curl $\vec{F} = 3$ everywhere in D but

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \frac{\langle -y,x\rangle}{x^2+y^2}\cdot d\vec{r} + \oint_C \frac{3}{2}\langle -y,x\rangle\cdot d\vec{r}$$

$$= \underbrace{2\pi}_{\text{was shown in class, C winds once around O.}} + \underbrace{3\,\text{area}(C)}_{\text{by Green's thm}}$$

$$= 2\pi + 3\cdot 4\pi = 2\pi + 12\pi = 14\pi.$$

(f) False. Correct is (angular speed of paddle wheel) $= \frac{1}{2}$ curl $\vec{F} = \frac{3}{2}$.

Problem 5. (4 points)
Compute the curl and the divergence of the vector field

$$\vec{F}(x,y,z) = xe^y\vec{\imath} + \sin(z)\cos(x)\vec{\jmath} + (x^2+y^2+z^2)\vec{k}$$

Curl:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xe^y & \sin z \cos x & x^2+y^2+z^2 \end{vmatrix}$$

$$= \langle \partial_y(x^2+y^2+z^2) - \partial_z(\sin z \cos x),\ \partial_z(xe^y) - \partial_x(x^2+y^2+z^2),\ \partial_x(\sin z \cos x) - \partial_y(xe^y) \rangle$$

$$= \langle 2y - \cos z \cos x,\ -2x,\ -\sin z \sin x - xe^y \rangle$$

[no points if your answer is a scalar]

div.

$$\vec{\nabla} \cdot \vec{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle xe^y, \sin z \cos x, x^2+y^2+z^2 \rangle$$

$$= \partial_x(xe^y) + \partial_y(\sin z \cos x) + \partial_z(x^2+y^2+z^2)$$

$$= e^y + 2z$$