Midterm Exam II

November 5, 2008

No books. No notes. No calculators. No electronic devices of any kind.

Name ____________________ **Student Number** ____________________

Problem 1. (5 points)

(a) Find a function $f(x, y, z)$ such that $\vec{\nabla} f = \vec{F}$, where $\vec{F}$ is the vector field $\vec{F}(x, y, z) = \langle yz + 2y, xz + 2x, xy + 6z \rangle$.

(b) Compute $\vec{\nabla} \times \vec{F}$.

(a) $\frac{\partial f}{\partial x} = yz + 2y$ $\quad f(x,y,z) = xyz + 2xy + C(y,z)$

$\frac{\partial f}{\partial y} = xz + 2x$ $\quad \frac{\partial f}{\partial y} = xz + 2x + \frac{\partial C}{\partial y}$ $\quad \frac{\partial C}{\partial y} = 0$

$C(y,z) = C(z)$.

$\frac{\partial f}{\partial z} = xy + 6z$ $\quad \frac{\partial f}{\partial z} = xy + C'(z)$ $\quad C'(z) = 6z$

$C(z) = 3z^2 + c$

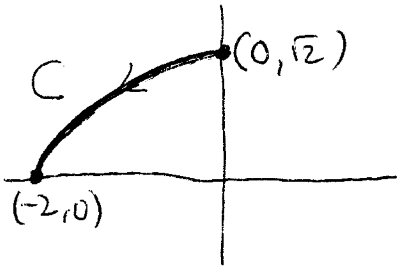
$\boxed{f(x,y,z) = xyz + 2xy + 3z^2 + c, \quad c \text{ constant.}}$

(b) in part (a) we showed that $\vec{F}$ is conservative. Therefore, the curl of $\vec{F}$ vanishes. $\boxed{\vec{\nabla} \times \vec{F} = \vec{0}}$.

1	2	3	4	5	6	total/30

Problem 2. (5 points)
Let C be the part of the parabola $y^2 = x + 2$ in the second quadrant, starting at the point $(0, \sqrt{2})$ and ending at the point $(-2, 0)$. Compute the line integral

$$\int_C \langle 5y, 2x \rangle \cdot d\vec{r}$$



parametrize C:

$x = t^2 - 2$

$y = t$ with t from $\sqrt{2}$ to 0.

(at the start point, $t = y = \sqrt{2}$, at end point $t = y = 0$.)

$$\int_C \langle 5y, 2x \rangle \cdot d\vec{r} = \int_C \langle 5y, 2x \rangle \cdot \langle dx, dy \rangle = \int_{\sqrt{2}}^{0} \langle 5t, 2t^2 - 4 \rangle \cdot \langle 2t, 1 \rangle \, dt$$

$$= \int_{\sqrt{2}}^{0} 10t^2 + 2t^2 - 4 \, dt = \int_{\sqrt{2}}^{0} 12t^2 - 4 \, dt = \left(4t^3 - 4t\right)\Big|_{\sqrt{2}}^{0}$$

$$= 0 - 0 - 4\sqrt{2}^3 + 4\sqrt{2} = -8\sqrt{2} + 4\sqrt{2} = \underline{\underline{-4\sqrt{2}}}$$

Another way to parametrize C is

$x = t$
$y = \sqrt{t+2}$ $\quad t$ from 0 to -2.

Because of the term $\sqrt{t+2}$, this is more work, but can be done:

$$\int_C \langle 5y, 2x\rangle \cdot d\vec{r} = \int_C \langle 5y, 2x\rangle \cdot \langle 1, \frac{1}{2\sqrt{t+2}}\rangle\, dt =$$

$$= \int_0^{-2} \langle 5\sqrt{t+2}, 2t\rangle \langle 1, \frac{1}{2\sqrt{t+2}}\rangle\, dt = \int_0^{-2} 5\sqrt{t+2} + \frac{t}{\sqrt{t+2}}\, dt$$

One way to solve this integral is with the substitution $u = \sqrt{t+2}$.

then $du = \frac{1}{2\sqrt{t+2}}\, dt$. Continue the calculation:

$$\ldots = \int_0^{-2} \frac{5(t+2) + t}{\sqrt{t+2}}\, dt = \int_0^{-2} \frac{12t+20}{2\sqrt{t+2}}\, dt = \int_{\sqrt{2}}^{0} 12(u^2-2)+20\, du$$

$$= \int_{\sqrt{2}}^{0} 12u^2 - 4\, du$$, which is the same integral as in the first solution.

$$= -4\sqrt{2}$$

Problem 3. (5 points)
Let C be the circle of radius 2 centered at the point $(1,1)$ in the xy-plane, oriented counterclockwise. Use Green's theorem to compute

$$\oint_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = \langle 5x - y + 2xy, 2x + x^2 + 3y \rangle$.

Let D be the disc bounded by C.

Green's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \, dA$$

$$\operatorname{curl} \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(2x + x^2 + 3y) - \frac{\partial}{\partial y}(5x - y + 2xy) = 2 + 2x - (-1 + 2x) = 3$$

So $$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \operatorname{curl} \vec{F} \, dA = \iint_D 3 \, dA = 3 \iint_D dA = 3 \operatorname{Area}(D) = 3(\pi 2^2) = \underline{\underline{12\pi}}$$

because the area of D is $\pi r^2 = \pi 2^2$ (circle of radius 2).

There is no need to set up an integral to compute the area of a circle of radius 2

Remark. Of course, the double integral is only so simple because the curl is constant. If you want to integrate a non-constant function over D in polar coordinates, the integral is:

$$\iint_D f(x,y) \, dA = \int_0^{2\pi} \int_0^2 f(1 + r\cos\theta, 1 + r\sin\theta) \, r \, dr \, d\theta$$

Problem 5. (5 points)
Compute the curl and the divergence of the vector field

$$\vec{F} = \langle \cos(xy), e^{z+y}, \tan(x+z) \rangle$$

Curl

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(xy) & e^{z+y} & \tan(x+z) \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}\tan(x+z) - \frac{\partial}{\partial z} e^{z+y}\right)$$

$$-\vec{j}\left(\frac{\partial}{\partial x}\tan(x+z) - \frac{\partial}{\partial z}\cos(xy)\right) + \vec{k}\left(\frac{\partial}{\partial x} e^{z+y} - \frac{\partial}{\partial y}\cos(xy)\right)$$

$$= \vec{i}\left(0 - e^{z+y}\right) - \vec{j}\left(\frac{1}{\cos^2(x+z)} - 0\right) + \vec{k}\left(0 + x\sin(xy)\right)$$

$$= \langle -e^{z+y}, -\frac{1}{\cos^2(x+z)}, x\sin(xy) \rangle$$

Divergence

$$\vec{\nabla} \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \langle \cos(xy), e^{z+y}, \tan(x+z) \rangle$$

$$= \frac{\partial}{\partial x}\cos(xy) + \frac{\partial}{\partial y} e^{z+y} + \frac{\partial}{\partial z}\tan(x+z)$$

$$= -y\sin(xy) + e^{z+y} + \frac{1}{\cos^2(x+z)}$$

Remark. It is very important that (for 3D vector fields) the curl is a **vector** and the divergence a **scalar**.

If you gave the curl as a scalar or the divergence as a vector: -2 pts.

Remark If you couldn't differentiate "tan" you lost 2 points. Sorry.

If you forget the formula, just work it out:

$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ so by the quotient rule:

$$\tan'(x) = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

Problem 6. (5 points)

(a) Verify that the vector field in the xy-plane given by $\vec{F} = \langle x^2, e^y \cos y \rangle$ is conservative.

(b) Let C be the half ellipse parametrized by

$$\vec{r}(t) = \langle 2\cos t, \sin t \rangle \qquad 0 \le t \le \pi \ .$$

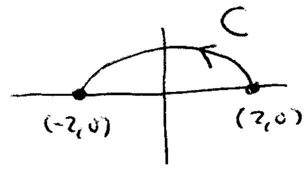
Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ using path independence.

(a) $\operatorname{curl} \vec{F} = \frac{\partial}{\partial x}(e^y \cos y) - \frac{\partial}{\partial y}(x^2) = 0 - 0 = 0.$

The domain of $\vec{F}$ is all of $\mathbb{R}^2$ (which is simply connected)

Therefore, $\vec{F}$ is conservative.

(b) $\vec{r}(0) = \langle 2, 0 \rangle$ $\quad \vec{r}(\pi) = \langle -2, 0 \rangle$



Because $\vec{F}$ is conservative, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path taken from (2,0) to (-2,0). You can replace C by any path you want, as long as the other path also starts at (2,0) and ends at (-2,0). Use the straight line along the x-axis:

$x = t$ $\quad t$: from 2 to -2.
$y = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \int_2^{-2} \langle t^2, e^0 \cos 0 \rangle \cdot \langle dt, 0 \rangle = \int_2^{-2} t^2 \, dt = \frac{1}{3} t^3 \Big|_2^{-2} = -\frac{8}{3} - \frac{8}{3} = -\frac{16}{3}$$

<u>Remark.</u> If you did not use this method of changing to a simpler path, you lost one point, I'm sorry.

[The potential function is $f(x,y) = \frac{x^3}{3} + \frac{1}{2} e^y (\cos y + \sin y)$. Using $f(x,y)$ the integration is $\int_C \vec{F} \cdot d\vec{r} = \left(\frac{x^3}{3} + \frac{1}{2} e^y (\cos y + \sin y) \right) \Big|_{(2,0)}^{(-2,0)} = -\frac{8}{3} + \frac{1}{2} - \frac{8}{3} - \frac{1}{2} = -\frac{16}{3}$.]