Midterm Exam I

October 2, 20123

No books. No notes. No calculators. No electronic devices of any kind.

Name ________________ **Student Number** ________________

Problem 1. (6 points)
The hyperboloid of one sheet with equation $x^2 + 4y^2 - z^2 = 16$ and the plane with equation $x + z = 2$ intersect in a curve C.

(a) Sketch the projection of C into the xy-plane.

(b) Find a parametric vector equation for the tangent line to C at the point $\langle 4, 1, -2\rangle$.

Hint: use $y = t$ to parametrize C.

Parametrize C: substitute $y=t$ into $x^2+4y^2-z^2=16$ to get $x^2+4t^2-z^2=16$. To solve for x in terms of t, use the second equation $x+z=2$ to eliminate z. Solve for z: $z=2-x$, substitute into $x^2+4t^2-z^2=16$ to get

$$x^2 + 4t^2 - (2-x)^2 = 16$$
$$x^2 + 4t^2 - 4 + 4x - x^2 = 16$$
$$4t^2 + 4x = 20$$
$$x = 5 - t^2$$

Substitute into $z=2-x$ to get $z = 2-(5-t^2) = t^2-3$.

So $\boxed{\vec{r}(t) = \langle 5-t^2, t, t^2-3\rangle}$ is a parametrization of C.

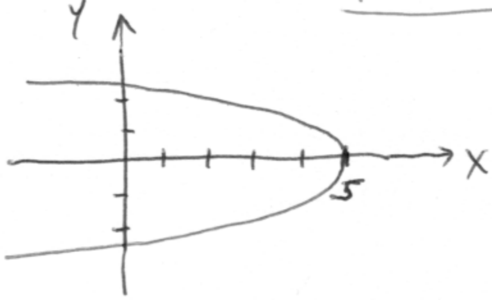
(continued)

1	2	3	4	5	total/21

Overflow space

(a) The xy-projection is parametrized by $\begin{cases} x = 5-t^2 \\ y = t \end{cases}$

and has equation $\boxed{x = 5 - y^2.}$ This is a parabola:



(b) $\vec{r}(t) = \langle 5-t^2, t, t^2-3 \rangle$

$\vec{r}'(t) = \langle -2t, 1, 2t \rangle$

$\vec{r}(1) = \langle 4, 1, -2 \rangle$ this is the point we want, $t = 1$.

$\vec{r}'(1) = \langle -2, 1, 2 \rangle$

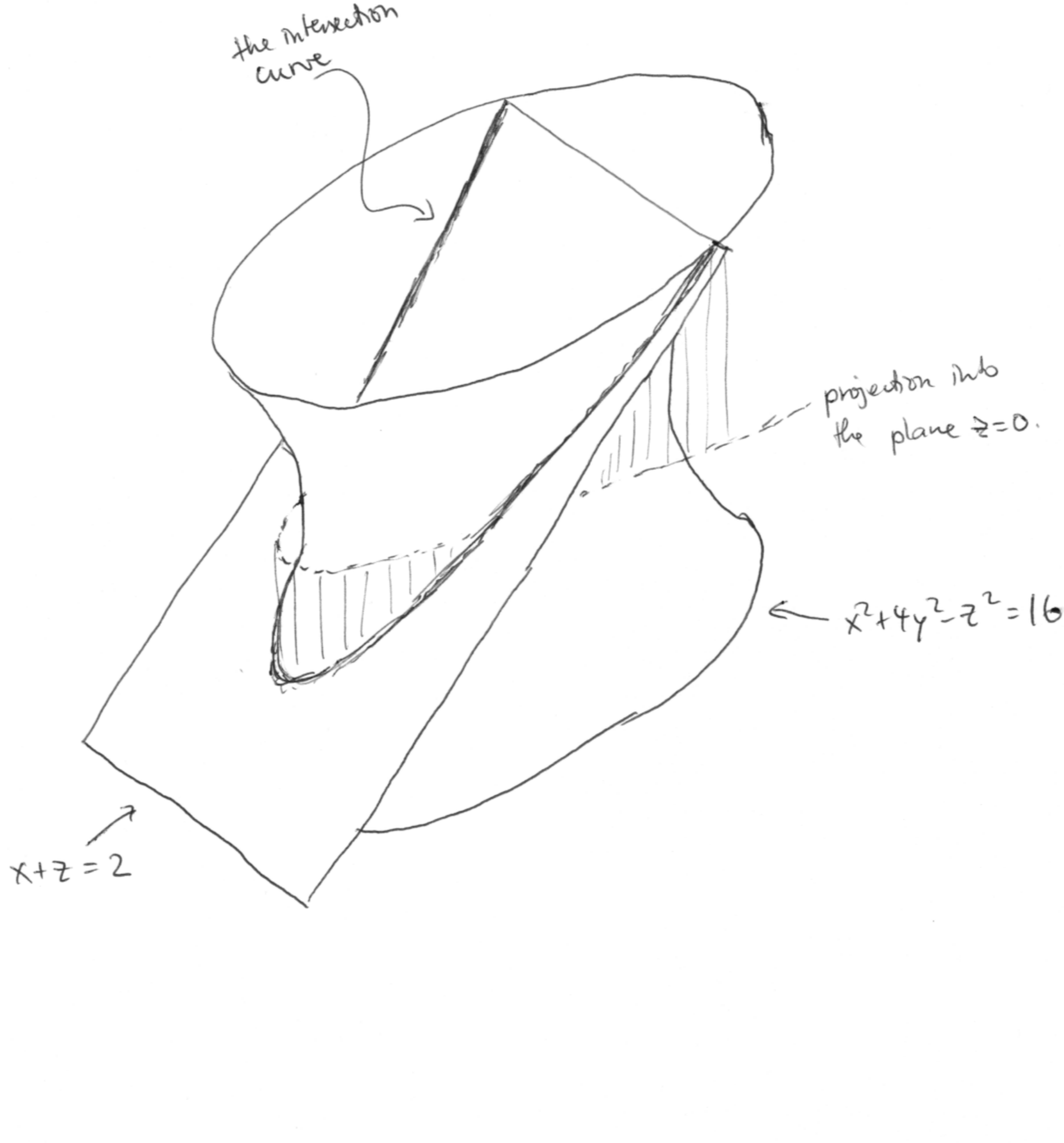
parametric vector equation for the tangent line:

$$\boxed{\langle 4,1,-2 \rangle + t\langle -2,1,2 \rangle}.$$

Remark. Finding the projection into the xy-plane does not mean setting $z=0$.

Setting $z=0$ gives the intersection with the xy-plane.

Eliminating z from the equations will give the projection.



Problem 2. (4 points)
The curve C is parametrized by the vector function

$$\vec{r}(t) = \langle t^3, 2, t^2 \rangle$$

Find the arc length of the part of C which is between the points $\langle 0, 2, 0 \rangle$ and $\langle 27, 2, 9 \rangle$.
Hint: the integral can be solved with a simple substitution.

$\vec{r} = \langle t^3, 2, t^2 \rangle$ on the interval $t \in [0, 3]$
b/c $\vec{r}(0) = \langle 0, 2, 0 \rangle$ and $\vec{r}(3) = \langle 27, 2, 9 \rangle$.

$\vec{r}\,' = \langle 3t^2, 0, 2t \rangle$

$\|\vec{r}\,'\| = \sqrt{9t^4 + 4t^2} = \sqrt{9t^2+4}\sqrt{t^2} = \sqrt{9t^2+4}\; t$ b/c $t \geq 0$ on the interval.

$ds = \|\vec{r}\,'\|\,dt = \sqrt{9t^2+4}\; t\, dt$

The arc-length is

$$\int_C ds = \int_0^3 \sqrt{9t^2+4}\; t\, dt \qquad u = 9t^2+4, \quad du = 18t\,dt$$

$$= \frac{1}{18}\int_4^{85} \sqrt{u}\, du = \frac{1}{18}\,\frac{2}{3}\,u^{3/2}\Big|_4^{85} = \frac{1}{27}\left(85^{3/2} - 4^{3/2}\right)$$

$$= \frac{1}{27}\left(85^{3/2} - 8\right)$$

$$= \frac{85\sqrt{85} - 8}{27}$$

Problem 3. (3 points)
True or false? (Assumme that a curve C is parametrized by a twice continuously differentiable vector function $\vec{r}(t)$, with velocity $\vec{v}(t)$, and acceleration $\vec{a}(t)$.)

(a) at a time t where $\kappa(t)$ reaches a maximum, we necessarely have $\vec{a}(t) \perp \vec{v}(t)$.
(b) at a time t where $\kappa(t)$ is zero, we must have that $\vec{a}(t) \parallel \vec{v}(t)$.
(c) Suppose that $\vec{r}(t) \parallel \vec{a}(t)$, for all t. Then the vector function $\vec{r}(t) \times \vec{v}(t)$ is constant, i.e., does not change in time.

(a) FALSE. (This is only true at constant speed. If the speed changes at the point of maximal curvature, $\vec{a}$ will have a non-zero tangential component, and will not be orthogonal to $\vec{v}$.)

(b) TRUE. (If $\kappa = 0$ the normal component of $\vec{a}$ is 0, so $\vec{a}$ is parallel to $\vec{v}$.)

(c) TRUE.

$$(\vec{r} \times \vec{v})' = \vec{r}\,' \times \vec{v} + \vec{r} \times \vec{v}\,'$$
$$= \underbrace{\vec{v} \times \vec{v}}_{\substack{=0 \\ \text{b/c } \vec{v} \parallel \vec{v}}} + \underbrace{\vec{r} \times \vec{a}}_{\substack{=0 \\ \text{b/c } \vec{r} \parallel \vec{a}}} = \vec{0}$$

so $\vec{r} \times \vec{v}$ is constant.

Problem 4. (4 points)
The curve C is parametrized by the vector funtion

$$\vec{r}(t) = \langle t^2, e^t, \cos t \rangle$$

Find equations for the osculating plane and the normal plane to C at the point corresponding to the parameter value $t = 0$.

$\vec{r} = \langle t^2, e^t, \cos t \rangle$

$\vec{r}' = \langle 2t, e^t, -\sin t \rangle$

$\vec{r}'' = \langle 2, e^t, -\cos t \rangle$

$\vec{r}(0) = \langle 0, 1, 1 \rangle$

$\vec{r}'(0) = \langle 0, 1, 0 \rangle$

$\vec{r}''(0) = \langle 2, 1, -1 \rangle$

Normal plane: normal vector is $\vec{r}'(0) = \langle 0, 1, 0 \rangle$

point on plane is $\vec{r}(0) = \langle 0, 1, 1 \rangle$

Eqn is $\langle 0,1,0 \rangle \cdot (\langle x, y, z \rangle - \langle 0,1,1 \rangle) = 0$

or $0(x-0) + 1(y-1) + 0(z-1) = 0$

or $\boxed{y = 1}$

Osculating plane: normal vector is $\vec{r}'(0) \times \vec{r}''(0) = \langle 0,1,0 \rangle \times \langle 2,1,-1 \rangle$

$= \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = \langle -1, 0, -2 \rangle$

point is $\vec{r}(0) = \langle 0,1,1 \rangle$

Eqn is $\langle -1, 0, -2 \rangle \cdot (\langle x, y, z \rangle - \langle 0,1,1 \rangle) = 0$

or $-1(x-0) + 0(y-1) - 2(z-1) = 0$

or $-x - 2z + 2 = 0$

or $\boxed{x + 2z = 2}$

Overflow space I.

Remark: At the point $\langle 0,1,1\rangle$ corresponding to $t=0$ we have:

$$\vec{T} = \langle 0,1,0\rangle$$

$$\vec{N} = \frac{\sqrt{5}}{5}\langle 2,0,-1\rangle$$

$$\vec{B} = \frac{\sqrt{5}}{5}\langle -1,0,-2\rangle$$

The quickest way to calculate these is:

(1) normalize $\vec{r}\,'(0)$ to get $\vec{T}$

(2) normalize $\vec{r}\,'(0)\times\vec{r}\,''(0)$ to get $\vec{B}$

(3) $\vec{N} = \vec{B}\times\vec{T}$.

This method avoids having to differentiate $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ to obtain $\vec{N}$.

But you do not need $\vec{T},\vec{N},\vec{B}$ for this problem.

Problem 5. (4 points)
The derivative $\frac{d}{dt}\Big(\vec{r}(t)\cdot\vec{r}'(t)\Big)$ is equal to

(a) $\vec{r}'(t)\cdot\vec{r}'(t)$,
(b) $\vec{r}(t)\cdot\vec{r}''(t)$,
(c) $\vec{r}'(t)\cdot\vec{r}''(t)$
(d) $||\vec{r}'(t)||^2+\vec{r}(t)\cdot\vec{r}''(t)$
(e) none of the above.

$$(\vec{r}\cdot\vec{r}')' = \vec{r}'\cdot\vec{r}' + \vec{r}\cdot\vec{r}''$$
$$= ||\vec{r}'||^2 + \vec{r}\cdot\vec{r}''$$

d is correct.

We have a smooth vector function $\vec{r}(t)$, with associated unit tangent vector $\vec{T}(t)$, principal normal vector $\vec{N}(t)$, and binormal vector $\vec{B}(t)$. The derivative $\frac{d}{dt}\vec{T}(t)$ is given by

(a) $\dfrac{\vec{r}'(t)\times\vec{r}''(t)}{||\vec{r}'(t)||^2}$,
(b) $\dfrac{||\vec{r}'(t)\times\vec{r}''(t)||}{||\vec{r}'(t)||^2}\,\vec{N}(t)$
(c) $\dfrac{||\vec{r}'(t)\times\vec{r}''(t)||}{||\vec{r}'(t)||^2}\,\vec{B}(t)$
(d) $\dfrac{||\vec{r}'(t)\times\vec{r}''(t)||}{||\vec{r}'(t)||^3}\,\vec{N}(t)$
(e) none of the above.

Start with the definition of curvature: $\dfrac{d\vec{T}}{dS} = \varkappa\vec{N}$ ($\varkappa$ = curvature)

Multiply by $\frac{ds}{dt}$ to get $\frac{d\vec{T}}{dt}$: $\dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}}{dS}\dfrac{ds}{dt} = \varkappa\dfrac{ds}{dt}\vec{N}$.

Recall the formulas $\varkappa = \dfrac{||\vec{r}'\times\vec{r}''||}{||\vec{r}'||^3}$ and $\dfrac{ds}{dt} = ||\vec{r}'||$:

$$\frac{d\vec{T}}{dt} = \frac{||\vec{r}'\times\vec{r}''||}{||\vec{r}'||^3}\,||\vec{r}'||\,\vec{N} = \frac{||\vec{r}'\times\vec{r}''||}{||\vec{r}'||^2}\,\vec{N}$$

b is correct.