Practice Midterm I

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (6 points)
The set of 2×3-matrices with entries in the field $\mathbb{F}$ is denoted by $M_{2\times 3}(\mathbb{F})$. (These matrices have 2 rows and 3 columns.) Such matrices have 6 entries, and we can think of 2×3-matrices as a different way of writing elements of $\mathbb{F}^6$. Therefore, we can turn $M_{2\times 3}(\mathbb{F})$ into an $\mathbb{F}$-vector space, by defining addition and scalar multiplication componentwise.

(i) Determine if
$$\{A \in M_{2\times 3}(\mathbb{F}) \mid A \text{ is in REF}\}$$
is a subspace of $M_{2\times 3}(\mathbb{F})$. Justify your answer.

(ii) Prove that
$$\begin{aligned}\{A \in M_{2\times 3}(\mathbb{F}) \mid\ & \\ & (1,1,1) \text{ is a solution of the homogeneous system} \\ & \text{of equations whose non-augmented coefficient matrix is } A\}\end{aligned}$$
is a subspace of $M_{2\times 3}(\mathbb{F})$ and find a basis for it.

Problem 2. (6 points)
Let V and W be subspaces of $\mathbb{R}^4$. Assume that $\dim V = 2$ and $\dim W = 3$.

(i) What are the possible dimensions of $V \cap W$?

(ii) For each of these dimensions, give explicit examples of V and W, where this dimension is achieved (in terms of spanning sets for V and W).

(iii) Explain why no other dimensions are possible.

Problem 3. (6 points)
Find out if the following vectors in $\mathbb{C}^4$ are linearly dependent. If they are, express one of them as a linear combination of the others.
$$\vec{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \end{pmatrix} \qquad \vec{v}_4 = \begin{pmatrix} -2 \\ -5 \\ 3 \\ -7 \end{pmatrix}$$

Problem 4. (6 points)

(a) Define the characteristic of a field.
(b) Does there exist a field with 6 elements? (Justify your answer)

Problem 5. (6 points)
Let $v_1, \dots, v_n$ be a linearly independent family of vectors in the $\mathbb{F}$-vector space V. Suppose that for every $v \in V$ the enlarged family $v, v_1, \dots, v_n$ is linearly dependent. Prove that V is spanned by $v_1, \dots, v_n$.

Problem 6. (6 points)
Find among the following list of vectors in $\mathbb{R}^3$ a linearly independent family

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix} \qquad \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 7. (6 points)
Extend the following list of vectors to a basis of $\mathbb{R}^4$.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 0 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 0 \\ 5 \\ -5 \\ -1 \end{pmatrix}$$

Problem 8. (6 points)
(a) Find the rank (i.e., the number of pivots in REF) of the matrix with coefficients in $\mathbb{R}$

$$\begin{pmatrix} 2 & 0 & 1 & 3 \\ 2 & 5 & 6 & 3 \\ 0 & 5 & 2 & 0 \end{pmatrix}$$

(b) Same matrix, but coefficients are in $\mathbb{F}_3$.

Problem 9. (6 points)
Four of King Arthurs knights sit around the round table, and are served unequal amounts of cereal. Every time the gong sounds, each knight takes half of the cereal from both of his neighbours. Determine the distribution of cereal in the knights bowls after 10 and 15 sounds of the gong.

Problem 10. (6 points)
The subspace $V \subset \mathbb{R}^4$ is spanned by $\left(\begin{smallmatrix}1\\0\\1\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1\\1\\0\\1\end{smallmatrix}\right)$. The subspace $W \subset \mathbb{R}^4$ is spanned by $\left(\begin{smallmatrix}2\\0\\2\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\\-1\\2\end{smallmatrix}\right)$. Find the dimension of $V \cap W$, and a basis for $V \cap W$.

Problem 11. (6 points)
What conditions do t and u have to satisfy in order for the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ t & 0 & 0 & 1 & 2 \\ u & 0 & 0 & t & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 & 0 \end{pmatrix}$$

to be invertible (i.e., have a pivot in every row and every column in REF)? (Assume $\mathbb{F} = \mathbb{Q}$, and $\mathbb{F} = \mathbb{F}_2$.)