# Practice Midterm II

**No books. No notes. No calculators. No electronic devices of any kind.**

**Problem 1.** (0 points)

(a) Find the general solution in $\mathbb{Q}^5$ of the system of equations

$$\begin{array}{ccccccccccc} x_1 & - & x_2 & - & 3x_3 & + & x_4 & & & = & 3 \\ & & x_2 & + & 2x_3 & + & x_4 & & & = & -1 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 2x_5 & = & 6 \\ x_1 & + & x_2 & + & x_3 & + & 3x_4 & + & x_5 & = & 4 \end{array}$$

(b) Note that the coefficient matrix $A \in M(4 \times 5, \mathbb{Q})$ of this system defines a linear map $A : \mathbb{Q}^5 \to \mathbb{Q}^4$. Find a basis for the image space of this linear transformation. What is the dimension of $\operatorname{im}(A)$?

(c) Find the coordinates of $\begin{pmatrix} 3 \\ -1 \\ 6 \\ 4 \end{pmatrix}$ with respect to the basis you found in (b).

**Problem 2.** (0 points)

Find the rank (i.e., the dimension of the image) and the nullity (i.e., the dimension of the kernel, or nullspace) of the orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto the vector $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Try to reason without calculating the matrix of $P$.

**Problem 3.** (0 points)

(a) Verify that $\mathcal{B} = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix})$ forms a basis of $\mathbb{R}^2$.

(b) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by $\pi/2$ around the origin. Find the matrix of $R$ with respect to the basis $\mathcal{B}$ of (a). In other words, find

$$M_{\mathcal{B}}^{\mathcal{B}}(R)\,.$$

**Problem 4.** (0 points)

For which choices of the constant $k \in \mathbb{Q}$ is the matrix

$$A_k = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$$

invertible?

**Problem 5.** (0 points)

Here the field is $\mathbb{F}_3$, the field with 3 elements. Consider the linear transformation

$$\begin{aligned} \phi : \mathbb{F}_3^2 &\longrightarrow \mathbb{F}_3^3 \\ (x, y) &\longmapsto (x + y, x - y, x)\,. \end{aligned}$$

(i) Find the matrix $M_{\mathcal{S}}^{\mathcal{S}}(\phi)$ of $\phi$ in standard bases of $\mathbb{F}_3^2$, $\mathbb{F}_3^3$.

(ii) Find a basis for the image of $\phi$.

(iii) Find all left inverses of $\phi$, i.e., all linear transformations $\psi : \mathbb{F}_3^3 \to \mathbb{F}_3^2$, such that $\psi \circ \phi = \mathrm{id}_{\mathbb{F}_3^2}$.

(iv) Why does $\phi$ not have any right inverses?

**Problem 6.** (0 points)
Evaluate the following determinant:

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix}$$

**Problem 7.** (5 points)
In $\mathbb{R}^3$ with the standard inner product, find a basis for the orthogonal complement of the line spanned by

$$\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

**Problem 8.** (0 points)
Consider the following matrix:

$$A = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & -2 \end{pmatrix}$$

The matrix $A$ is orthogonal, and has determinant 1. Therefore, $A$ describes a rotation about an axis through the origin.

(i) Find the rotation axis.

(ii) Find the rotation angle $\theta$, where $0 \leq \theta \leq \pi$.

**Problem 9.** (0 points)
The standard matrix of a linear transformation $T$ is

$$[T]_{\mathcal{E}} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}.$$

Find a basis $\mathcal{F}$ of $\mathbb{R}^2$, such that

$$[T]_{\mathcal{F}} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} .$$

**Problem 10.** (0 points)
Find the determinant of the matrix

$$\begin{pmatrix} 3 & 5 & 5 & 1 \\ 4 & 5 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 4 & 4 & 2 & 1 \end{pmatrix} .$$

**Problem 11.** (0 points)
Find all values of $x$ for which the matrix

$$\begin{pmatrix} 3 & 5 & 0 \\ 5 & 0 & x \\ 0 & 3 & 3 \end{pmatrix}$$

is invertible.

**Problem 12.** (0 points)
Find an invertible $3 \times 3$ matrix $E$, such that $EA$ is in reduced row echelon form, where

$$A = \begin{pmatrix} 1 & 5 & 0 & -2 \\ 2 & 10 & -2 & 0 \\ 3 & 15 & -1 & 1 \end{pmatrix} .$$

**Problem 13.** (0 points)
Consider the $\mathbb{R}$-vector space $P_n$ of polynomials of degree at most $n$ over $\mathbb{R}$. Differentiation defines an endomorphism $D : P_n \to P_n$. Find the determinant of $D$.

**Problem 14.** (0 points)
Left multiplication by the matrix $A \in M(n \times n, \mathbb{F})$ defines an endomorphism of $M(n \times n, \mathbb{F})$. Find the determinant of this endomorphism.

**Problem 15.** (0 points)

(a) Find the standard matrix of the orthogonal projection onto the line $2x_1+x_2 = 0$ in $\mathbb{R}^2$.

(b) Suppose $u_1, \ldots, u_k$ is an orthonormal system in $\mathbb{R}^n$, and $U$ the subspace it generates. Let $A$ be the $n \times k$-matrix whose columns are $u_1, \ldots, u_n$. Express the standard matrix of the projection onto $U$ in terms of $A$.

**Problem 16.** (0 points)
Let $V$ be a Euclidean vector space and $U \subset V$ a subspace. Denote the projection onto $U$ by $P_U$. The *reflection across* $U$ is the linear map $R_U : V \to V$ defined by the formula $R_U = 2P_U - \mathrm{id}_V$. Find a formula for the determinant of $R_U$ in terms of $\dim V$ and $\dim U$.

**Problem 17.** (0 points)
Let $Ax = b$ be an inhomogeneous system of linear equations over $\mathbb{R}$. Suppose $b$ is orthogonal to the columns of $A$. Prove that the system has no solutions, unless $b = 0$.

**Problem 18.** (0 points)
Let $(V, \langle , \rangle)$ be a finite-dimensional Euclidean vector space. Prove that there is an orthogonal isomorphism $V \to \mathbb{R}^n$, for some $n$. (Here $\mathbb{R}^n$ is endowed with the standard inner product).