Midterm II

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (8 points)

(i) Define the term **linear map**.
(ii) Define the term **kernel of a linear map**.
(iii) Give an example of a map $f : \mathbb{R}^2 \to \mathbb{R}^3$, which is not linear.
(iv) Give an example of a linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$, whose kernel had dimension 2.
(v) Prove that if $f : V \to W$ is a linear map, then the kernel of f is a subspace of V.
(vi) Is it possible for a linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ to be injective? Justify your answer.

Problem 2. (8 points)
Find the standard matrix of the reflection across the line with equation $3x - 4y = 0$ in $\mathbb{R}^2$.

Problem 3. (8 points)

(i) Define the determinant of an $n \times n$ matrix with coefficients in a field $\mathbb{F}$.
(ii) Calculate the determinant of

$$\begin{pmatrix} 2 & 3 & x & 0 \\ 4 & 0 & 0 & y \\ -1 & z & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where $x, y, z \in \mathbb{F}$.

Problem 4. (8 points)

(i) Define the terms **eigenvalue**, **eigenvector**, and **eigenspace**.
(ii) Prove that if a linear map $P : V \to V$ satisfies $P^2 = P$, then every eigenvalue of P is either equal to 0 or equal to 1.
(iii) Prove that every linear map $P : V \to V$ which satisfies $P^2 = P$ is diagonalizable. **Hint:** write an arbitrary vector $v \in V$ as $v = P(v) + \big(v - P(v)\big)$.

Problem 5. (8 points)
Find formulas for x_n, y_n, given that

$$x_{n+1} = -5x_n + 4y_n\,, \qquad y_{n+1} = -12x_n + 9y_n\,,$$

and

$$x_0 = 1\,, \qquad y_0 = 1\,.$$

Compute $\lim_{n\to\infty} \frac{x_n}{y_n}$ and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$.

Problem 1.

(i) A map $f: V \to W$ (V, W vector spaces over the field $\mathbb{F}$) is **linear** if $f(\lambda v + w) = \lambda f(v) + f(w)$ $\forall v, w \in V, \lambda \in \mathbb{F}$.

(ii) the **kernel** of the linear map $f: V \to W$ is

$$\ker(f) = \{ v \in V \mid f(v) = 0 \}.$$

(iii) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is not linear.

(iv) $f: \mathbb{R}^3 \to \mathbb{R}^2$ with matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has kernel of dimension 2.

(v) Let $f: V \to W$ be linear. **Claim:** $\ker(f) \subset V$ is a subspace.

Pf. ① $\ker f \neq \emptyset$ because $f(0) = 0$ and so $0 \in \ker f$.

② $v, w \in \ker(f) \Rightarrow f(v) = 0$ & $f(w) = 0$

$\Rightarrow f(v+w) = f(v) + f(w) = 0$

$\Rightarrow v + w \in \ker f$ so $\ker f$ closed under add.

③ $v \in \ker(f), \lambda \in \mathbb{F} \Rightarrow f(v) = 0$

$\Rightarrow \lambda f(v) = \lambda 0 = 0$

$\Rightarrow f(\lambda v) = 0$

$\Rightarrow \lambda w \in \ker(f)$ so $\ker f$ closed under scalar mult.

(vi) Suppose $f: \mathbb{R}^3 \to \mathbb{R}^2$ is injective. Then $\ker(f) = \{0\}$.

So $\dim(\ker f) = 0$. Dimension formula: $\dim(\ker f) + \dim(\operatorname{im} f) = 3$.

So $\underline{\dim(\operatorname{im} f) = 3}$. But $\operatorname{im}(f) \subset \mathbb{R}^2$ is a subspace, so $\underline{\dim(\operatorname{im} f) \leq 2}$

contradiction. f cannot be injective.

Problem 2.

$\begin{pmatrix}4\\3\end{pmatrix}$ is a vector on the reflection mirror $3x-4y=0$.

So $\begin{pmatrix}4\\3\end{pmatrix}$ is an eigenvector with eigenvalue 1 for the reflection across $3x-4y=0$.

$\begin{pmatrix}3\\-4\end{pmatrix}$ is orthogonal to the reflection mirror, so it is an eigenvector with eigenvalue -1 for this reflection.

~~the~~ So $\begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}3\\-4\end{pmatrix}$ is a basis consisting of eigenvectors.

$$\boxed{A = PDP^{-1}}$$

The reflection is diagonalizable. Use the formula in the box.

P: change of basis matrix: has basis vectors as columns:

$$P = \begin{pmatrix}4 & 3\\3 & -4\end{pmatrix}$$

$D = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ has the eigenvalues on the diagonal.

$$P^{-1} = \frac{1}{-16-9}\begin{pmatrix}-4 & -3\\-3 & 4\end{pmatrix} = \frac{1}{-25}\begin{pmatrix}-4 & -3\\-3 & 4\end{pmatrix} = \frac{1}{25}\begin{pmatrix}4 & 3\\3 & -4\end{pmatrix}.$$

$$A = \begin{pmatrix}4 & 3\\3 & -4\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\frac{1}{25}\begin{pmatrix}4 & 3\\3 & -4\end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix}4 & -3\\3 & 4\end{pmatrix}\begin{pmatrix}4 & 3\\3 & -4\end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix}16-9 & 12+12\\12+12 & 9-16\end{pmatrix} = \frac{1}{25}\begin{pmatrix}7 & 24\\24 & -7\end{pmatrix} = \frac{1}{25}\begin{pmatrix}7 & 24\\24 & -7\end{pmatrix}$$

Problem 3.

(i) $\det : M(n\times n, \mathbb{F}) \longrightarrow \mathbb{F}$ is the unique function wich satisfies:

- det is linear in each row
- det is zero for matrices of rk $< n$
- $\det I_n = 1$.

(ii) $$\begin{vmatrix} 2 & 3 & x & 0 \\ 4 & 0 & 0 & y \\ -1 & z & 2 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}_{\text{expand}} = \begin{vmatrix} 2 & x & 0 \\ 4 & 0 & y \\ -1 & 2 & 0 \end{vmatrix}_{\text{expand}} = -y \begin{vmatrix} 2 & x \\ -1 & 2 \end{vmatrix} = -y(4+x).$$

Problem 4.

(i) Let $f : V \to V$ be an endomorphism of the $\mathbb{F}$-vectorspace V.
$\lambda \in \mathbb{F}$ is an eigenvalue of f if there exists a non-zero $v \in V$ s.t. $f(v) = \lambda v$,
$v \in V$ is an eigenvector of f if $v \neq 0$ and there exists a $\lambda \in \mathbb{F}$ s.t. $f(v) = \lambda v$,
eigenspace of eigenvalue λ is $\ker(\lambda\, id_V - f) =: E_\lambda$

(ii) Suppose λ is an eigenvalue of P. Then $P^2(v) = P(Pv) = P(\lambda v) = \lambda P(v) = \lambda^2 v$.
Also, $P^2(v) = P(v) = \lambda v$.
Hence, $\lambda^2 v = \lambda v \Rightarrow (\lambda^2 - \lambda) v = 0 \overset{v \neq 0}{\Rightarrow} \lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0$
$\Rightarrow \lambda = 0$ or $\lambda = 1$.

(iii) Write $v \in V$ as $v = P(v) + v - P(v)$.
then $P(v) \in E_1$ since $P(P(v)) = P(v)$
and $v - P(v) \in E_0$ since $P(v - P(v)) = P(v) - P^2(v) = P(v) - P(v) = 0$.
So $V = E_1 + E_0$. Putting a basis of E_1 together with a basis for E_2 gives a basis of V consisting of eigenvectors for P.

Problem 5.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ -12 & 9 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\begin{vmatrix} -5-\lambda & 4 \\ -12 & 9-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 45 + 48 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3). \quad 2$$

$\lambda = 3$: $\quad E_3 = \ker \begin{pmatrix} -5-3 & 4 \\ -12 & 9-3 \end{pmatrix} = \ker \begin{pmatrix} -8 & 4 \\ -12 & 6 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = 1$: $\quad E_1 = \ker \begin{pmatrix} -5-1 & 4 \\ -12 & 9-1 \end{pmatrix} = \ker \begin{pmatrix} -6 & 4 \\ -12 & 8 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad 2$

General solution:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 3^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 1^n \begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 3^n \\ 2\cdot 3^n \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Find c_1, c_2:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & +1 \end{array}\right)$$

$c_1 = -1, \; c_2 = +1.$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = -\begin{pmatrix} 3^n \\ 2\cdot 3^n \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 - 3^n \\ 3 - 2\cdot 3^n \end{pmatrix} \qquad \boxed{\begin{aligned} x_n &= 2 - 3^n \\ y_n &= 3 - 2\cdot 3^n \end{aligned}} \quad 2$$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{2-3^n}{3-2\cdot 3^n} = \lim_{n\to\infty} \frac{2/3^n - 1}{3/3^n - 2} = \frac{0-1}{0-2} = \frac{1}{2} \quad \text{(slope of } E_3\text{)}$$

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$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim \frac{2-3^{n+1}}{2-3^n} = \lim_{n\to\infty} \frac{2/3^n - 3}{2/3^n - 1} = \frac{0-3}{0-1} = 3. \quad \text{(larger EV.)}$$

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