Midterm I

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (5 points)
What conditions do u, v, x, y, z have to satisfy for the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ u & v & x & y & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

to be in

(i) row echelon form,

(ii) reduced row echelon form.

(i) the pivot in the second row has to be in column 3 or 4
(otherwise the pivots don't conform to the echelon pattern)
so $u=0$ and $v=0$ and (one of x, y not $=0$).
z arbitrary.

(ii) Now, in addition the pivots have to $=1$
and entries above the pivots have to be zero.
So the pivot in the second row has to be in column 3.
So $u=0$, $v=0$, $x=1$, $z=0$ and y arbitrary.

<u>Note:</u> If you are allowed row operations, you can put <u>any</u> matrix in REF and RREF, so the problem would not make sense.

The answer changes if you swap rows, for example.
For $\begin{pmatrix} u & v & x & y & z \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ the answers would be

(i) $u \neq 0$ all others arbitrary

(ii) $u=1$, $v=0$, $z=0$, x, y arbitrary.

Problem 2. (6 points)

(i) Define what it means for a family of vectors $v_1, \dots, v_n \in V$, for an $\mathbb{R}$-vector space V, to be linearly independent.

(ii) What conditions does u have to satisfy in order for the three vectors

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -1 \\ u \\ 1 \end{pmatrix}$$

to be linearly independent in $\mathbb{R}^3$?

(i) $(v_1, \dots, v_n)$ is linearly independent if the only solution to $\lambda_1 v_1 + \dots + \lambda_n v_n$ $(\lambda_1, \dots, \lambda_n \in \mathbb{R})$ is $\lambda_1 = 0, \dots, \lambda_n = 0$.

(ii) To find the linear relations among these vectors solve the homogeneous system

$$x_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ u \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Put the coeff. matrix in REF:

$$\begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & u \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & u \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} +I \sim \begin{pmatrix} 1 & 1 & u \\ 0 & 2 & -1 \\ 0 & 0 & 1+u \end{pmatrix}.$$

To have only the trivial solution, need a pivot in every column, so need $1+u \neq 0$ or $u \neq -1$.

So (v_1, v_2, v_3) is linearly independent if and only if $u \neq -1$.

Problem 3. (6 points)
Consider the $\mathbb{R}$-vector space $\mathrm{Fun}([-1,1],\mathbb{R})$. Decide if the following are subspaces. If yes, give a proof, if no, explain why not.

(i) $A = \{f : [-1,1] \to \mathbb{R} \mid f \text{ is even, i.e., } \forall x \in [-1,1]\colon f(x) = f(-x)\}$

(ii) $B = \{f : [-1,1] \to \mathbb{R} \mid f(0) = 1\}$

(i) A is a subspace:

(1) The zero vector in $\mathrm{Fun}([-1,1],\mathbb{R})$ is the zero function, given by $0(x) = 0$ $\forall x \in [-1,1]$. This function is even:
$0(x) = 0 = 0(-x)$ $\forall x \in [-1,1]$. So $0 \in A$ and A is non empty.

(2) If $f, g \in A$ then, for all $x \in [-1,1]$ we have

$$(f+g)(x) = f(x) + g(x) \quad \text{(defn of "+" in } \mathrm{Fun}([-1,1],\mathbb{R})\text{)}$$
$$= f(-x) + g(-x) \quad \text{since } f, g \in A$$
$$= (f+g)(-x) \quad \text{(defn of "+" in } \mathrm{Fun}([-1,1],\mathbb{R})\text{)}$$

So $f+g \in A$.

(3) If $f \in A$ and $\lambda \in \mathbb{R}$, then for $x \in [-1,1]$ we have

$$(\lambda f)(x) = \lambda f(x) \quad \text{(defn of scalar mult in } \mathrm{Fun}([-1,1],\mathbb{R})\text{)}$$
$$= \lambda f(-x) \quad \text{since } f \text{ is even}$$
$$= (\lambda f)(-x) \quad \text{(defn of scalar mult in } \mathrm{Fun}([-1,1],\mathbb{R})\text{)}$$

So λf is also even.

(ii) B is not a subspace since B does not contain the zero vector in $\mathrm{Fun}([-1,1],\mathbb{R})$. The 0-function is not in B:
$0(0) = 0 \neq 1$.

Problem 4. (5 points)
Find among the following vectors in $\mathbb{R}^3$ a basis of $\mathbb{R}^3$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

You may assume that these vectors span $\mathbb{R}^3$.

To find among a spanning ~~set a linearly ind~~ family a basis, we need to find a maximally independent subfamily. So we need the linear relations among v_1, v_2, v_3, v_4. Proceed as in 2(ii):

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 1 \\ -1 & 3 & -4 & 1 \end{pmatrix} \begin{matrix} \\ -2\mathrm{I} \\ +\mathrm{I} \end{matrix} \sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -5 & 5 & 1 \\ 0 & 5 & -5 & 1 \end{pmatrix} \begin{matrix} \\ \\ +\mathrm{II} \end{matrix} \sim \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -5 & 5 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

We see that (v_1, v_2, v_4) is linearly independent because the pivots are in columns 1, 2, 4.

Also, v_3 is a linear combination of v_1, v_2.

So (v_1, v_2, v_4) is maximally independent and hence a basis of $\mathrm{span}(v_1, v_2, v_3, v_4)$ (which is $\mathbb{R}^3$ by information given ~~fro~~) ~~Bro~~

(Note that since $\dim \mathbb{R}^3 = 3$, any linearly independent family of 3 vectors is a basis of $\mathbb{R}^3$. So this proves that (v_1, v_2, v_4) is a basis of $\mathbb{R}^3$ without referring to the information given.)

Problem 5. (6 points)

(i) Define the term **basis** of a vector space V over the field $\mathbb{F}$.

(ii) Find a basis for the intersection of the two subspaces $V, W \subset \mathbb{R}^3$.

$$V = \operatorname{span}(\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\2\\3\end{pmatrix}) \qquad W = \operatorname{span}(\begin{pmatrix}-1\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}).$$

(i) A family $(v_1, \dots, v_n)$ of vectors in V is a basis of V if ① $(v_1, \dots, v_n)$ is linearly independant and ② $V = \operatorname{Span}(v_1, \dots, v_n)$.

(ii) Every vector in V is of the form $a\begin{pmatrix}1\\1\\2\end{pmatrix} + b\begin{pmatrix}2\\2\\3\end{pmatrix}$ and every vector in W is of the form $c\begin{pmatrix}-1\\0\\2\end{pmatrix} + d\begin{pmatrix}0\\1\\3\end{pmatrix}$.

~~So $V \cap W = \{$~~ To find $V \cap W$ set these equal:

$a\begin{pmatrix}1\\1\\2\end{pmatrix} + b\begin{pmatrix}2\\2\\3\end{pmatrix} = c\begin{pmatrix}-1\\0\\2\end{pmatrix} + d\begin{pmatrix}0\\1\\3\end{pmatrix}$. This is a homogeneous system of 3 equations with indeterminants a, b, c, d. to solve it put the coefficient matrix in RREF:

$$\begin{pmatrix}1 & 2 & 1 & 0\\ 1 & 2 & 0 & -1\\ 2 & 3 & -2 & -3\end{pmatrix}\begin{matrix}\\ -\mathrm{I}\\ -2\mathrm{I}\end{matrix} \sim \begin{pmatrix}1 & 2 & 1 & 0\\ 0 & 0 & -1 & -1\\ 0 & -1 & -4 & -3\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 1 & 0\\ 0 & 1 & 4 & 3\\ 0 & 0 & 1 & 1\end{pmatrix}\begin{matrix}-\mathrm{III}\\ -4\mathrm{III}\\ \\ \end{matrix}$$

$$\sim \begin{pmatrix}1 & 2 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{pmatrix}\begin{matrix}-2\mathrm{II}\\ \\ \\ \end{matrix} \sim \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{pmatrix}.$$

General solution is

$a = -d$
$b = d$
$c = -d$
$d = d$

or $\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = d\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$.

So $a\begin{pmatrix}1\\1\\2\end{pmatrix} + b\begin{pmatrix}2\\2\\3\end{pmatrix} = -d\begin{pmatrix}1\\1\\2\end{pmatrix} + d\begin{pmatrix}2\\2\\3\end{pmatrix} = d\begin{pmatrix}1\\1\\1\end{pmatrix}$.

So $V \cap W = \operatorname{span}\begin{pmatrix}1\\1\\1\end{pmatrix}$ a basis is given by the family consisting of the single vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$.