Sylvester Inertia Theorem

The field is $\mathbb{R}$.

V: finite dim'l $\mathbb{R}$-VS.

$q: V \longrightarrow \mathbb{R}$ a quadratic form.

Then there exists a basis $B = (v_1, \dots, v_n)$ for V

such that

$$\begin{tikzcd} V \arrow[r, "q"] & \mathbb{R} \\ \mathbb{R}^n \arrow[u, "\Phi_B"] \arrow[ur, "Q_A"'] & \end{tikzcd}$$

where A is the matrix of q w.r.t. B.

$$Q_A(x) = \underset{1\times n}{x^t}\ \underset{n\times n}{A}\ \underset{n\times 1}{x}.$$

the matrix $A = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & 0 & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & 0 & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$

$$A = \left(\begin{array}{c|c|c} I_r & 0 & 0 \\ \hline 0 & -I_s & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

$$Q_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_{r+s}^2 .$$

(no cross terms)

<u>Proof.</u> Existence: last time.

Rmk: the proof gives a method to finding a Sylvester basis.

uniqueness of r, s.

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, "\Psi"'] \arrow[dr, "\boxed{Q_B}"] & \\ \boxed{V \arrow[r, "q"] & \mathbb{R}} \\ \mathbb{R}^n \arrow[u, "\Phi"] \arrow[ur, "\boxed{Q_A}"'] & \end{tikzcd}$$

how are B, A related?

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "Q_B"] & \mathbb{R} \\ \mathbb{R}^n \arrow[u, "\underbrace{\Psi^{-1}\circ\Phi}_{=P}"] \arrow[ur, "Q_A"'] & \end{tikzcd}$$

inv. matrix.

We show that r = max dimension of a subspace $U \subset V$ s.t. $q|U$ is positive definite.

Rmk. ~~If Take~~ let $(v_1, \dots, v_n)$ be a Sylvester basis for q.

~~Let~~ $W = \text{span}(v_1, \dots, v_r)$.

the matrix of $q|W$ wrt basis $(v_1, \dots, v_r)$

is identity matrix so

$$q\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underline{x_1^2 + \dots + x_n^2}$$ standard inner product.

this is positive definite:

$q|W$ is pos. def. ~~and~~ $r = \dim W$.

so max dim of a pos. def. subspace is $\geq r$.

Assume $U \subset V$ is a subspace s.t. $q|U$ pos. def.

Assume $\dim U > r$.

By the dimension formula: $U \cap \text{span}(v_{r+1}, \dots, v_n)$

~~$\dim \geq r$~~ $\dim > r$ $\quad\quad$ $\dim = n - r$.

$\neq \{0\}$.

$(\dim(U+W) + \dim U \cap W = \dim U + \dim W)$

so if $0 \neq v \in U \cap \underline{\text{span}(v_{r+1}, \dots, v_n)}$ then

$q(v,v) =$

on $\mathrm{Span}(v_{r+1}, \dots, v_n)$ q has matrix

$$\begin{pmatrix} -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

$$q\begin{pmatrix} x_{r+1} \\ \vdots \\ x_n \end{pmatrix} = -x_{r+1}^2 \cdots - x_{r+s}^2 \leq 0 .$$

So $q(v,v) \leq 0$. So $q|U$ is not positive definite.

Similarly, s = max dim. of a subspace on which q is negative definite.

(i.e. ~~$q(v,v)$~~ $q(v) < 0$ $\forall$ $v \neq 0$.)

Sylvester done $\square$.

Consider two symmetric matrices to be "equivalent" if ~~there~~ "their quadratic forms differ by a coordinate transformation" iow if $\exists$ ~~$P \in GL(n$~~ $P \in GL(n, \mathbb{R})$ s.t.

$$A = P^t B P.$$

~~Sylvester~~: for short A, B "symmetrically equivalent"

Sylvester: Every symmetric real matrix is symmetrically equivalent to a unique matrix in Sylvester normal form.

If A and B are two matrices representing the quadratic form w.r.t. two different bases then ~~we have~~. there exists an inv. $n \times n$ matrix P s.t.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "Q_B"] & \mathbb{R} \\ \mathbb{R}^n \arrow[u, "P"] \arrow[ur, "Q_A"'] & \end{tikzcd}$$

commutes.

$$Q_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = Q_B\left(P\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right)$$

$$Q_A(x) = Q_B(Px)$$

$$\boxed{x^t \underline{A} x} = (Px)^t BPx = \boxed{x^t \underline{P^t B P} x}$$

Hence $\boxed{A = P^t B P}$

Aside: bil. form det. by quadratic form.

$y^t A x = y^t P^t B P x$ $\forall x, y$. In partic. for e_i e_j:

$e_i^t A e_j = e_i^t P^t B P e_j$

$a_{ij} = (P^t B P)_{ij}$.

• Similarity of matrices: A sim B if $\exists P$ inv. s.t.

$$A = P^{-1} B P.$$

(A, B square matrices)

over $\mathbb{C}$: Jordan canonical form.

• orthogonal similarity of symmetric matrices:

A, B symmetric, real. $A \sim_{\text{orthsim}} B$ if $\exists P$ orth

$$A = P^{-1} B P \qquad \boxed{P^{-1} = P^t}$$

spectral theorem / principal axes thm.

• symmetric equivalence of symm. matrices: $A \sim B$ $\exists P$ inv.

$$A = P^t B P.$$

Sylvester thm.

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to discuss properties of $q(x) = 1$. (conic sections in $\mathbb{R}^2$.)

$q(x)=1$

$2x_1^2 + 3x_2^2 = 1$

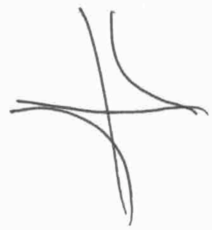
$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $x_1^2 + x_2^2 = 1$

should take orthogonal coordinate transformations to preserve lengths / angles.

in $\mathbb{R}^2$

q is pos. def.



$x_1^2 - x_2^2 = 1$.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

signature 0.

Methods for finding Sylvester bases:

- orthogonally diagonalize the ~~A~~ symmetric matrix A. get an ON basis w.r.t $\langle , \rangle$.
  q is diagonal in this ON basis.
  $\to$ rescale vectors to get Sylvester basis.

- If q pos. definite: Gram-Schmidt.

- ~~inductive~~ recursive procedure following the proof of Sylvester.

- symmetric transformations.
  A symm. matrix, P change of basis matrix that diagonalizes A makes A into a Sylvester matrix:

$$S = P^t A P \quad \text{Also} \quad P = IP$$

symmetric operations on A, the corresponding column operations on I. to get S and P.

e.g. $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} - \frac{1}{2} I$ $\qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 0 & 5/2 \end{pmatrix}$ $\qquad \downarrow$

$\begin{pmatrix} 2 & 0 \\ 0 & 5/2 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} \frac{2}{\sqrt{2}} & 0 \\ 0 & 5/2 \end{pmatrix}$ $\qquad \downarrow$

$\begin{pmatrix} 1 & 0 \\ 0 & 5/2 \end{pmatrix}$ $\qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$

$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2}\frac{\sqrt{2}}{\sqrt{5}} \\ 0 & \frac{\sqrt{2}}{\sqrt{5}} \end{pmatrix}$.

$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}\frac{\sqrt{2}}{\sqrt{5}} \\ \frac{\sqrt{2}}{\sqrt{5}} \end{pmatrix}$ Sylvester basis.

A pos. definite.

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\quad q\begin{pmatrix} x \\ y \end{pmatrix} = 2xy$. $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

swapping doesn't work.

adding:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

procede.

$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad P = \begin{pmatrix} \sqrt{2}/2 & -\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$.