Jordan Canonical form.

Jordan block matrix

$$k\left\{\underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{k}\right. = J_\lambda^{(k)}$$

A matrix is in JCF if it is block diagonal with Jordan blocks along the diagonal.

$$\begin{pmatrix} \boxed{J} & 0 & 0 & 0 \\ 0 & \boxed{J} & 0 & 0 \\ 0 & 0 & \boxed{J} & 0 \\ 0 & 0 & 0 & \boxed{J} \end{pmatrix}.$$

<u>Thm.</u> ① Every $A \in M(n\times n, \mathbb{C})$ is similar to a matrix in JCF

② A sim to B $\Longleftrightarrow$ JCF agree.

(up to reordering of the blocks)

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \underset{\text{sim.}}{\sim} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

A sim B iff. $\exists\ P \in GL(n, \mathbb{C})$.

$= $ invertible $n \times n$ matrices.

s.t. $\boxed{A = PBP^{-1}}$.

If $f: V \to V$

$\mathcal{B}, \mathcal{C}$ different bases of V

then $[f]_{\mathcal{B}}$ sim $[f]_{\mathcal{C}}$.

Question: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim_{\mathbb{R}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\sim$ similar.

Remark: if $A, B \in M(n \times n, \mathbb{R})$

then $A \sim_{\mathbb{C}} B \iff A \sim_{\mathbb{R}} B$.

Proof. "$\Leftarrow$" trivial. $\exists P$: $A = PBP^{-1}$.

"$\Rightarrow$" Suppose $\underline{A = PBP^{-1}}$ where $P \in GL_n(\mathbb{C})$.

$P = R + iS$ where $R, S \in M(n \times n, \mathbb{R})$.

P invertible, but we don't know if R or S are invertible.

We have:

$$AP = PB$$

$$\underbrace{AR}_{} + i\underbrace{AS}_{} = \underbrace{RB}_{} + i\underbrace{SB}_{\text{real}}.$$

$$\left.\begin{aligned} AR &= RB \\ AS &= SB \end{aligned}\right\} \text{not done yet.}$$

Trick: Consider the polynomial function

$$\det(R + tS) \in \mathbb{R}[t].$$
$$\in \mathbb{C}[t].$$

if $t = i$ $\underline{\det(R+tS)} = \det(R+iS)$

$= \det(P)$

$\neq 0$ b/c P invertible.

the polynomial $\det(R+tS)$ is not zero.

So there exist $\lambda \in \mathbb{R}$ s.t.

$$\det(R+\lambda S) \neq 0.$$

then $R+\lambda S \in GL(n,\mathbb{R})$.

$$AR + \lambda AS = RB + \lambda SB$$

$$A(R+\lambda S) = \underbrace{(R+\lambda S)}_{\in GL(n,\mathbb{R})} B.$$

So $A \sim_{\mathbb{R}} B$. $\square$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$ JCF?

only eigenvalue is 1.

alg. mult. 3.

geom. multiplicity:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ $\dim E_1 = 1$ geom. mult is 1.

so JCF: $$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\text{In JCF Because for every eigenvalue the geom. mult. = \# Jordan blocks.}}$

$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\dim E_1 = 1$.

1 Jordan block

JCF $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \operatorname{sim}_{\mathbb{R}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Our tools allow us to compute the algebraic multiplicities ($\leftarrow$ if we can decompose the char poly into linear factors) and geom. multiplicities.

$\Rightarrow$ for each λ the # of Jordan blocks.

for alg mult up to 3 this lets us det. the JCF.

4×4: <u>geom. mult = 2</u>.
$$\begin{array}{cc} 1 & 3 \end{array} \quad \begin{array}{cc} 2 & 2 \end{array}$$
$$\left(\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline & \lambda & 1 & 0 \\ & & \lambda & 1 \\ & & & \lambda \end{array}\right) \quad \left(\begin{array}{cc|cc} \lambda & 1 & 0 & 0 \\ & \lambda & 0 & 0 \\ \hline & & \lambda & 1 \\ & & & \lambda \end{array}\right).$$

Classification of quadratic forms over $\mathbb{R}$.

Sylvester Normal form.

Sylvester Inertia theorem

Recall: V : real vector space.

$\beta : V \times V \longrightarrow \mathbb{R}$ symm. bil. form.

$q(v) = \beta(v,v)$ the associated quadratic form.

the matrix A given by

$$a_{ij} = \beta(v_i, v_j)$$

is the matrix of q w.r.t. the basis $\mathcal{B} = (v_1, \dots, v_n)$.

$$[q]_{\mathcal{B}} = A.$$

$$\begin{tikzcd} V \arrow[r, "q"] & \mathbb{R} \\ \mathbb{R}^n \arrow[u, "\Phi_{\mathcal{B}}"] \arrow[ur, "Q_A"'] & \end{tikzcd}$$

Q_A : quadratic form on $\mathbb{R}^n$ given by A.

$$Q_A(x) = x^t A x .$$

$$Q_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1 \dots x_n) A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum a_{ij} x_i x_j .$$

<u>Definition</u>. A basis $B = (v_1, \dots, v_n)$ s.t.

$$[q]_B = \left(\begin{array}{ccc|cccccc} +1 & & & & & & & & \\ & \ddots & & & & 0 & & & \\ & & +1 & & & & & & \\ \hline & & & -1 & & & & & \\ & 0 & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{array} \right) \begin{array}{l} \}\, r \\ \\ \}\, s \\ \\ \end{array} \Bigg\}\, n .$$

i.e.

$$q(x_1 v_1 + \dots + x_n v_n) = x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_{r+s}^2 .$$

is called a <u>Sylvester basis</u> for q.

<u>Example</u>. If q, β is ~~a~~ positive definite. then β is a Euclid. inner product.

then ~~a~~ every ON basis. is a Sylvester basis.

$$\beta(v_i, v_j) = \delta_{ij}$$

$$[\beta] = I_n . \quad \begin{array}{l} r = n \\ s = 0 . \end{array}$$

<u>Thm</u> (Sylvester Inertia theorem).

For a quadratic form on an n-dim'l real VS V there always exists a Sylvester basis. The numbers r, s are independent of the choice of the Syl. basis.

$r+s$ is the <u>rank</u> of q

$r-s$ is the <u>signature</u> of q.

<u>Proof.</u> Induction on n.

If $q=0$ then every basis is a Sylvester basis.
$(r=0,\ s=0)$.

Assume $q \neq 0$. Then there exists a vector $v \in V$ s.t. $q(v) \neq 0$. Then of course $v \neq 0$.

Consider
$$U = \{ w \in V \mid \beta(v,w) = 0 \}$$
(If β inner product $U = (\operatorname{span} v)^{\perp}$.).

Consider $\beta(v,\cdot) : V \longrightarrow \mathbb{R}$.

$U =$ ker of this linear map.

the lin map $\beta(v,\cdot)$ is surjective. (b,c $\beta(v,v) = q(v) \neq 0$.)

By dimension formula,

$\dim U = n-1$.

By induction hypothesis there exist a Sylvester basis for $(U, q|U)$.

~~then~~ $v_1, \dots, v_{n-1}$, then $q(v_i) = \pm 1$.

$\beta(v_i, v_j) = 0 \quad i \neq j$.

$\beta(v, v_i) = 0$.

$q(v) = \beta(v,v) \neq 0$.

by ~~multiply~~ rescaling v:

$$q(\lambda v) = \lambda^2 q(v) \overset{?}{=} \pm 1$$

$q(v) = \frac{1}{\lambda^2}$

we can make $\boxed{q(\lambda v) = \pm 1.}$

$\lambda^2 = \frac{1}{q(v)}$

$\lambda^2 q(v) = \pm 1$.

$\lambda^2 = \pm \frac{1}{q(v)}$

$$\lambda = \sqrt{\left|\frac{1}{q(v)}\right|} .$$

we have $q(v) = \pm 1$

then we have a Sylvester basis.

Existence done.