Equivalence Relations.

Recall: A relation on a set X, denoted $\sim$

$(R \subset X \times X,\ x \sim y$ means $(x,y) \in R)$

is an equivalence relation if
- reflexive $x \sim x \ \forall x \in X$.
- symmetric, $x \sim y \Rightarrow y \sim x$
- transitive: $x \sim y,\ y \sim z \Longrightarrow x \sim z$.

For $x \in X$ we have

$[x] = \{ y \in X \mid x \sim y \}$ .. equivalence class of $x \in X$.

If $\sim$ is an equivalence relation then:

- Every $x \in X$ is contained in an equivalence class:

  $x \in [x]$ b/c $\sim$ reflexive.

- $\cancel{[x]\cap[y]=}$ If $[x],[y]$ equivalence classes

  either $[x] \cap [y] = \emptyset$

  or $[x] = [y]$.

  if $z \in [x]$ and $z \in [y]$ then $[x] = [y]$.

So X decomposes into equivalence classes.

Example. $X = M(m \times n, \mathbb{F})$.

relation: row equivalence.

$A \sim_R B$ if and only if there exists an invertible $P \in M(m \times m, \mathbb{F})$:
$$PA = B.$$

① $A \sim_R B$ if and only if $\mathrm{RREF}(A) = \mathrm{RREF}(B)$.

② Every equivalence class contains a unique matrix in RREF.

Example: $X = M(m \times n, \mathbb{F})$.

$A \sim B$ equivalent if there exist $Q \in M(n \times n, \mathbb{F})$ and $P \in M(m \times m, \mathbb{F})$ both invertible, such that

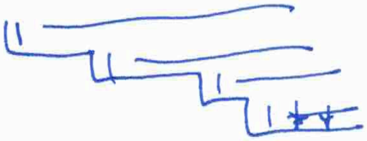
$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "A"] & \mathbb{F}^m \\
\mathbb{F}^n \arrow[r, "B"] \arrow[u, "Q"] & \mathbb{F}^m \arrow[u, "P"']
\end{tikzcd}$$

$$B = P^{-1} A Q.$$

or $\boxed{A = P B Q^{-1}}$

$A \sim B$ if and only if A can be converted into B by a sequence of elementary row & column operations.

Every matrix $A \sim \left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}^{\mathrm{rk}\, A} & 0 \\ \hline 0 & 0 \end{array}\right).$



Theorem: ① $A \sim B$ if and only if $\mathrm{rk}\, A = \mathrm{rk}\, B$.

② ~~Every equivalence class~~ ~~there are~~

$$\left(\; 0 \; \right), \quad \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right), \quad \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right), \ldots \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right)$$

complete list of equivalence classes

until $k = \min(m,n)$.

Version for linear maps:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \Phi \uparrow & & \uparrow \Psi \\ V' & \xrightarrow{g} & W' \end{array}$$

$f \sim g$ if $\exists$ isomorphisms Φ, Ψ as in the diagram. Then $f \sim g \iff \mathrm{rk} f = \mathrm{rk}\, g$.

Moreover for every linear map $f: V \to W$

$$\begin{tikzcd} V \arrow[r, "f"] & W \\ \mathbb{F}^n \arrow[u, "\Phi"] \arrow[r, "A"] & \mathbb{F}^m \arrow[u, "\Psi"'] \end{tikzcd}$$

there exist bases of V and W such that the matrix of f is

$$\left(\begin{array}{c|c} I_{\mathrm{rk} f} & 0 \\ \hline 0 & 0 \end{array}\right).$$

<u>Proof</u>. choose a basis of $\ker(f)$ $\qquad r = \mathrm{rk} f.$

$w_1 \dots w_{n-r}$ $\qquad \dim V = n.$

complete to a basis of V.

$v_1 \dots v_r$.

$f(v_1) \dots f(v_r)$ are lin indep in W

complete to a basis $y_{r+1} \dots y_m$.

$$\begin{array}{c|ccc|c} & v_1 & \dots & v_r & w_1 \dots w_{n-r} \\ \hline f(v_1) & 1 & 0 & 0 & \\ & 0 & 1 & \vdots & 0 \\ \vdots & \vdots & & & \\ f(v_r) & 0 & 0 & 1 & \\ \hline y_{r+1} & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & 0 \\ y_m & 0 & 0 & 0 & \end{array}$$

Specialize to endomorphisms

$f: V \to V$.

$$\begin{tikzcd} V \arrow[r, "f"] & V \\ \mathbb{F}^n \arrow[u, "\Phi"] \arrow[r, "A"] & \mathbb{F}^n \arrow[u, "\Phi"'] \end{tikzcd}$$

Matrix version:

$B, A \in M(n\times n, \mathbb{F})$ are <u>similar</u> if

there exists an invertible matrix $P \in M(n\times n, \mathbb{F})$ such that $PB \cancel{A} = AP$

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "A"] & \mathbb{F}^n \\ \mathbb{F}^n \arrow[u, "P"] \arrow[r, "B"] & \mathbb{F}^n \arrow[u, "P"'] \end{tikzcd}$$

$$\boxed{A \text{ sim } B \iff A = PBP^{-1}}$$

quite hard to decide ~~..~~ when $A \sim B$ or to describe all similarity classes.

<u>Rmk:</u> If $A \sim B$ then $\text{charpoly}_A = \text{charpoly}_B$.

(also $\det A = \det B$.) $\quad A = PBP^{-1}$

$$\det(A - tI) = \det(PBP^{-1} - tI)$$

$$= \det(PBP^{-1} - PtIP^{-1}) = \det(P(B-tI)P^{-1})$$

$= \cancel{\det(P)} \det(B - tI) \cancel{\det(P)^{-1}}$

$= \det(B - tI)$.

Jordan canonical form

Answers the question how to describe all similarity classes over $\mathbb{F} = \mathbb{C}$.

Let $\lambda \in \mathbb{C}$. $J_\lambda^{(k)} = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ Jordan Block Matrix.

(λ) 1×1 Jordan block

$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ 2×2

3×3

$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

etc.

char poly of Jordan block

$\begin{vmatrix} \lambda - t & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda - t \end{vmatrix} = (\lambda - t)^k$

k = size of Jordan block

λ only eigenvalue.

algebraic multiplicity k.

Eigenspace: $\mathrm{Nul} \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$ RREF.

$\dim E_\lambda = 1$.

<u>Theorem.</u> Every $A \in M(n\times n, \mathbb{C})$ is similar to ~~exactly one matrix~~ a matrix in <u>Jordan canonical form</u>, i.e.

Block diagonal, with Jordan blocks on diagonal.

$$\begin{pmatrix} \boxed{J_{\lambda_1}^{(k_1)}} & & & 0 \\ & \boxed{J_{\lambda_2}^{(k_2)}} & & \\ & & \ddots & \\ 0 & & & \boxed{J_{\lambda_r}^{(k_r)}} \end{pmatrix}$$

The Jordan canonical form is unique up to reordering the Jordan blocks.

~~For~~ Let $f: V \to V$ be an endom of a $\mathbb{C}$-VS V.

Then $\exists$ basis B of V s.t.

$$[f]_B \text{ is in JCF.}$$