Quadratic Forms

Application of the spectral theorem to quadratic forms.

Let V be a vector space over a field $\mathbb{F}$, char $\mathbb{F} \neq 2$.

$2 = 1+1 \neq 0$.

this means that $\frac{1}{2} \in \mathbb{F}$.

Let $\beta : V \times V \longrightarrow \mathbb{F}$ be a symmetric bilinear form.

$\forall v, w \in V$: $\beta(v, w) \in \mathbb{F}$.

symmetry: $\beta(v,w) = \beta(w,v)$.

The associated quadratic form q of β:

$$q : V \longrightarrow \mathbb{F}$$
$$v \longmapsto q(v) = \beta(v,v).$$

(For example, if $\mathbb{F} = \mathbb{R}$, $\beta = \langle , \rangle$ an inner product

then assoc. quadratic form is

$q(v) = \langle v, v \rangle = \|v\|^2$.)

Example. In $V = \mathbb{F}^n$ then

Every symmetric bilinear form is given by a symmetric matrix.

$\beta : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ bilinear.

$[\beta] = (\beta(e_i, e_j))_{ij}$ matrix of β. symmetric matrix.

Conversely $A \in M(n \times n, \mathbb{F})$ symmetric.

get $\beta(x, y) = x^t A y \in \mathbb{F}$.

e.g. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ symmetric matrix.

$$\beta(x, y) = (x_1 \; x_2 \; x_3) \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= \cancel{x_1 y_1 + 4 x_1 y_2 + 2 x_1 y_3 + 4 y_1^2}$$

$$= (x_1 \; x_2 \; x_3) \begin{pmatrix} y_1 + 2y_2 + y_3 \\ 2y_1 + 4y_2 + 0 \\ y_1 + 0 + 3y_3 \end{pmatrix}$$

$$= x_1 y_1 + \underline{2 x_1 y_2} + x_1 y_3$$
$$+ \underline{2 x_2 y_1} + 4 x_2 y_2 + 0$$
$$x_3 y_1 + 0 + 3 x_3 y_3 .$$

The associated quadratic form:

$$q(x) = q\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \beta\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)$$

$$= x_1^2 + 4x_1x_2 + 2x_1x_3 + 0x_2x_3$$
$$+ 4x_2^2 + 3x_3^2 .$$

every term is quadratic in the components of the vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

~~Conversely~~

In general, if $A = (a_{ij})_{ij}$ is the matrix of the bilinear form β. Then

$$q\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1 \cdots x_n)\, A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum_{ij} x_i a_{ij} x_j$$

$$= \sum_i a_{ii} x_i^2 + \sum_{i<j} x_i a_{ij} x_j + \sum_{i>j} x_i a_{ij} x_j$$

$$= \sum_i a_{ii} x_i^2 + \sum_{i<j} x_i a_{ij} x_j + \sum_{j>i} x_j a_{ji} x_i$$

$$= \sum_i a_{ii} x_i^2 + 2 \sum_{i<j} x_i a_{ij} x_j$$ b/c A symmetric.

$$\boxed{q\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_i \underline{\underline{a_{ii}}}\, x_i^2 + \sum_{i<j} 2a_{ij}\, x_i x_j}$$

to get from q to A:

$$q\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_i a_i x_i^2 + \sum_{i<j} b_{ij} x_i x_j$$

$$a_{ii} = a_i$$

$$a_{ij} = \tfrac{1}{2} b_{ij}$$

$$a_{ji} = \tfrac{1}{2} b_{ij}$$

for example:

$$q\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 4x_2^2 + 3x_3^2 + \underline{\underline{4x_1x_2}} + 2x_1x_3 + 0x_2x_3$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix} \rightsquigarrow \beta.$$

The symmetric bilinear form β can be reconstructed from q by

$$\beta(v,w) = \frac{1}{2}\left(q(v+w) - q(v) - q(w)\right).$$

Check:

$$\beta(v,w) = \frac{1}{2}\left(\beta(v+w, v+w) - \beta(v,v) - \beta(w,w)\right)$$

is true for every S.B.F.

If char $\mathbb{F} \neq 2$ we have a 1:1 correspondence between ~~quadrat~~ symm. bilinear forms and their associated quadratic forms.

Now let $(V, \langle , \rangle)$ be a Euclidean vector space.

Given a selfadjoint endomorphism $f: V \to V$

$(* \ \langle v, fw \rangle = \langle fv, w \rangle)$

define a corresponding symmetric bilinear form

$\boxed{\beta(v,w) = \langle v, fw \rangle}$ bilinear.

symmetric: $\underline{\beta(v,w)} = \langle v, fw \rangle$

$= \langle fv, w \rangle$ f self-adj

$= \langle w, fv \rangle$ $\langle , \rangle$ symm.

$= \underline{\beta(w,v)}$ defn of β.

Conversely, for every bilinear form $\beta: V \times V \to \mathbb{R}$ there exists a unique self-adjoint endomorphism $f: V \to V$ such that

$\beta(v,w) = \langle v, fw \rangle \quad \forall v, w \in V.$

Proof. Let β be given. Let (u_i) be an ON basis of V.

Let $A = [\beta]_{(u_1 \dots u_n)}$.

$$a_{ij} = \beta(u_i, u_j)$$

then let f: be the endomorphism s.t. $[f]_{(u_1 \dots u_n)} = A$.

then: $a_{ij} = \langle u_i, fu_j \rangle$.

since $(u_1 \dots u_n)$ is an ON basis f self-adjoint b/c A is symmetric, since β is symmetric.

Claim: $\langle v, fw \rangle = \langle \sum_i v_i u_i, f \sum_j w_j u_j \rangle$

$$= \sum_{ij} v_i w_j \underbrace{\langle u_i, fu_j \rangle}_{= a_{ij}}$$

$$= \sum_{ij} v_i w_j \beta(u_i, u_j)$$

$$= \beta\left(\sum_i v_i u_i, \sum_j w_j u_j\right)$$

$$= \beta(v, w). \quad \square$$

$V, \langle , \rangle$ Euclidean VS
$(u_1 \dots u_n)$ an ON basis then

$$\begin{tikzcd}
\boxed{\text{quadratic forms}} \arrow[r, leftrightarrow, "1:1"] & \text{symm. bil. forms on } V \arrow[r, leftrightarrow, "1:1"] \arrow[d, leftrightarrow, "1:1"] & \text{self-adj. endos of } V \arrow[dl, leftrightarrow, "1:1"] \\
& \text{symmetric matrices in } M(n \times n, \mathbb{R}) &
\end{tikzcd}$$

The spectral theorem says that for every quadratic form $q : V \to \mathbb{R}$. there exists an ON basis $u_1, \dots, u_n$ of V. such that

$$q(v) = q\,[v]_B = q\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2$$

w/o <u>cross terms</u>.

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the associated endomorphism/matrix.

Example. $\mathbb{R}^2$.

Consider the quadratic form

$$q\begin{pmatrix} x \\ y \end{pmatrix} = \frac{11}{4}x^2 - \frac{1}{2}\sqrt{3}\,xy + \frac{9}{4}y^2 .$$

~~Defi~~ defines a curve in $\mathbb{R}^2$ $\quad q\begin{pmatrix} x \\ y \end{pmatrix} = 1$.

$$\boxed{\frac{11}{4}x^2 - \frac{1}{2}\sqrt{3}\,xy + \frac{9}{4}y^2 = 1}$$

(conic section).

the corresponding symmetric matrix:

$$A = \begin{pmatrix} \frac{11}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{9}{4} \end{pmatrix}.$$

diagonalize A: $\begin{vmatrix} \frac{11}{4} - \lambda & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{9}{4} - \lambda \end{vmatrix}$

$$= \left(\frac{11}{4} - \lambda\right)\left(\frac{9}{4} - \lambda\right) - \frac{1}{16}3$$

$$= \lambda^2 - 5\lambda + \frac{99}{16} - \frac{3}{16} = \lambda^2 - 5\lambda + \frac{96}{16} = \lambda^2 - 5\lambda + 6$$

$$= (\lambda - 2)(\lambda - 3)$$

Eigenvalues 2, 3.

$$\lambda = 2 \qquad \begin{pmatrix} \frac{11}{4} - 2 & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{9}{4} - 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} \sim \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

Nullspace spanned by $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$. $\quad \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\lambda = 3: \quad \begin{pmatrix} \frac{11}{4} - 3 & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{9}{4} - 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & -\frac{3}{4} \end{pmatrix} \sim \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix}$$

nullspace spanned by $\cancel{\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}}$. $\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$.

$\cancel{P = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}}$

Basis of eigenvectors:

$V_1 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad V_2 \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$.

$V_1 \perp V_2 = 0. \quad \sqrt{3} - \sqrt{3} = 0.$

<u>Normalize:</u> $\|v_1\| = \sqrt{1 + \sqrt{3}^2} = \sqrt{4} = 2$

$\|v_2\| = 2$.

An ON basis of Eigenvectors:

$$u_1 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \rightsquigarrow \text{EV } 2.$$

$$u_2 = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \rightsquigarrow \text{EV } 3.$$

Diagonalization:

$$\begin{pmatrix} \frac{11}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{9}{4} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

q in the new coordinate system given by the ON basis: $\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$, $\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$.

$$\boxed{q\begin{pmatrix} x' \\ y' \end{pmatrix} = 2x'^2 + 3y'^2}.$$

$$\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}. \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

[illegible]

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\boxed{\begin{aligned} x &= \tfrac{1}{2}x' + \tfrac{1}{2}\sqrt{3}\,y' \\ y &= \tfrac{1}{2}\sqrt{3}\,x' - \tfrac{1}{2}y' \end{aligned}}$$

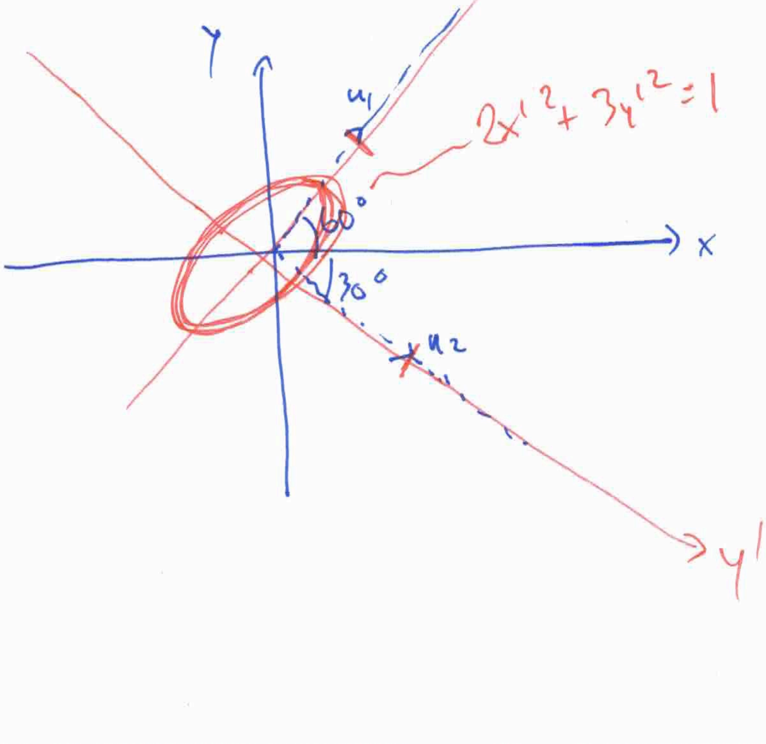
substitute this into

$$\boxed{\frac{11}{4}x^2 - \frac{1}{2}\sqrt{3}\,xy + \frac{9}{4}y^2 = 1}$$

new eqn $\boxed{2x'^2 + 3y'^2 = 1}$.

$$u_1 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

$$u_2 = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$$



$$\left(\frac{x'}{1/\sqrt{2}}\right)^2 + \left(\frac{y'}{1/\sqrt{3}}\right)^2 = 1.$$

ellipse w/

$\frac{1}{\sqrt{2}}$ $\quad \frac{1}{\sqrt{3}}$

The eigenspaces of the symmetric matrix corr. to the quadratic form q are the principal axes of $q(x)=1$.

eigenvalues: ~~[illegible]~~.