Self Adjoint Operators and the Spectral Theorem

(1) Motivation: Quantum Mechanics. (baby version).

State Space: Euclidean vector space.

$(V, \langle\rangle)$. states $\psi \in V$.

$\|\psi\|^2 = 1$.

Dynamical system:

$$\psi_{n+1} = U\psi_n \qquad U: V \longrightarrow V$$

Axiom: U: orthogonal operator. $\langle U\varphi, U\psi\rangle = \langle\varphi, \psi\rangle$

Observables: selfadjoint operators. $A: V \to V$.

you can measure only eigenvalues of Observables.

$$\langle A\varphi, \psi\rangle = \langle\varphi, A\psi\rangle.$$

After measurement the state is an eigenvector corr. to the eigenvalue which was measured.

Theorem: Selfadjoint operators are orthogonally diagonalizable:

$\psi \in V$: ~~$\psi = \sum \psi$~~

$$\psi = \sum_{\substack{\lambda:\,EV\\ \text{of } A.}} \psi_\lambda \qquad V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_n}$$

$\lambda_1 \dots \lambda_n$ eigenvalues of A.

$\lambda \neq \mu$: $\psi_\lambda \perp \psi_\mu$

Pythagoras $1 = \|\psi\|^2 = \sum_\lambda \|\psi_\lambda\|^2$.

Q.M: If the system is in state $\psi \in V$

you perform measurement A.

the propability that you measure λ.

is $\|\psi_\lambda\|^2$.

In ~~the~~ actual QM:

ground field is $\mathbb{C}$ not $\mathbb{R}$.

V: $\mathbb{C}$ - vector space.

$\langle , \rangle$: Hermitian inner product.

$$\langle \psi, \varphi \rangle = \overline{\langle \varphi, \psi \rangle}$$

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$\forall \varphi \in V$: $\langle \varphi, \varphi \rangle \in \mathbb{R}$.

$\langle \varphi, \varphi \rangle > 0$ makes sense.

Orthogonal $\longrightarrow$ "unitary"

$$\langle U\varphi, U\psi \rangle = \langle \varphi, \psi \rangle.$$

Self-adjoint: $\boxed{\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle.}$ ← no complex conjugation there! Sorry for the error!!

$\Rightarrow$ eigenvalues of self-adj. operators are $\in \mathbb{R}$.

Another difference: $\dim V = \infty$.

$\sim$ "Hilbert space".

Spectral Theorem / Principal Axes theorem.

Let V be a Euclidean vector space.

Let $f: V \longrightarrow V$ be a self-adjoint lin. map.

(1) Let λ, μ be eigenvalues of f. $\lambda \neq \mu$.
with corresponding eigenvectors v, w.

Claim: $v \perp w$.

Proof. $f(v) = \lambda v$
$f(w) = \mu w$.

$$\langle v, f(w) \rangle = \langle v, \mu w \rangle = \mu \langle v, w \rangle$$
$$\langle v, f(w) \rangle = \langle f(v), w \rangle = \langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

subtract:

$$0 = \underbrace{(\mu - \lambda)}_{\neq 0} \langle v, w \rangle. \quad \Rightarrow \quad \langle v, w \rangle = 0$$

so $v \perp w$.

(2) Assume $W \subset V$ subspace such that $f(W) \subset W$. Then Claim: $f(W^\perp) \subset W^\perp$.

(2) Assume $f(W) \subset W$.

<u>Claim:</u> $f(W^\perp) \subset W^\perp$.

Let $v \in W^\perp$. <u>Claim:</u> $f(v) \in W^\perp$.

Let $w \in W$ arbitrary.

<u>Claim:</u> $f(v) \perp w$.

$\Leftrightarrow \langle f(v), w \rangle = 0$

$\Leftrightarrow \langle v, \underbrace{f(w)}_{\in W} \rangle = 0$.

indeed true b/c $v \in W^\perp$. $\square$

(3) Every self-adjoint operator <u>$f: V \to V$</u> has an eigenvector.

Use the <u>fundamental theorem of algebra:</u>

* Every polynomial with complex coefficients has a root in $\mathbb{C}$.

Char poly of f is $\in \mathbb{R}[t]$.

has a root $\lambda + i\omega \in \mathbb{C}$.

$\lambda, \omega \in \mathbb{R}$.

Write the corresponding eigenvector as

$$v + iw = \begin{pmatrix} v_1 + iw_1 \\ \vdots \\ v_n + iw_n \end{pmatrix} \in \mathbb{C}^n.$$

Instead of dealing with $f : V \to V$

choose an ON basis of V and

consider $[f]_{\mathcal{B}} = A$ symmetric matrix.

$A : \mathbb{R}^n \to \mathbb{R}^n$ the corr. self. adj. operator on $(\mathbb{R}^n, \text{std})$

$$\begin{tikzcd} V \arrow[loop right, "f"] \\ \mathbb{R}^n \arrow[u, "\Phi"] \arrow[loop right, "A"] \end{tikzcd}$$

to prove that f is diagonalizable

suffices to prove A

is diagonalizable.

$A : \mathbb{R}^n \to \mathbb{R}^n$.

$\boxed{A : \mathbb{C}^n \to \mathbb{C}^n.}$

consider $\mathbb{R} \subset \mathbb{C}$: $\underset{\mathbb{R}\text{-VS}}{\mathbb{R}^n} \subset \underset{\mathbb{C}\text{-VS}}{\mathbb{C}^n}$. subset.

We have.

$$\underbrace{A(v+iw)}_{\in \mathbb{C}^n} = \underbrace{(\lambda+i\omega)}_{\in \mathbb{C}}\underbrace{(v+iw)}_{\in \mathbb{C}^n}$$

$$\underbrace{Av}_{\in \mathbb{R}^n} + i\underbrace{Aw}_{\in \mathbb{R}^n} = \underbrace{\lambda v - \omega w}_{\in \mathbb{R}^n} + i(\underbrace{\lambda w + \omega v}_{\in \mathbb{R}^n}) \quad \text{in } \mathbb{C}^n.$$

$$\Rightarrow \quad Av = \lambda v - \omega w \qquad Aw = \lambda w + \omega v$$

$$\boxed{\begin{aligned} Av &= \lambda v - \cancel{\omega w} \\ Aw &= \cancel{\omega v} + \lambda w \end{aligned}} \quad v, w \in \mathbb{R}^n$$

$$A(\text{span}(v,w)) \subset \text{span}(v,w).$$

A symmetric:

$$\langle Av, w\rangle = \langle v, Aw\rangle$$

$$\langle \lambda v - \omega w, w\rangle = \langle v, \omega v + \lambda w\rangle$$

$$\lambda\cancel{\langle v,w\rangle} - \omega\langle w,w\rangle = \omega\langle v,v\rangle + \lambda\cancel{\langle v,w\rangle}$$

$$\omega(\underbrace{\|v\|^2 + \|w\|^2}_{\neq 0.}) = 0 \quad . \quad \text{in } \mathbb{R}$$

Since $v+iw$ is an eigenvector, it is non-zero.

so $v \neq 0$ or $w \neq 0$.

so $\|v\|^2 > 0$ or $\|w\|^2 > 0 \Rightarrow \|v\|^2 + \|w\|^2 > 0$.

Conclusion: A symmetric $\Rightarrow \omega = 0$.

If $v \neq 0$: $Av = \lambda v$ shows that v is an eigenvector.

Finally 3

If $w \neq 0$: $Aw = \lambda w$ ~~…~~ w is an EV. is proved!

Proof of Spectral theorem. using 2 3

~~…~~ Remarks:

Every subspace of a Euclidean vector space is Euclidean.

If $f: V \to V$ is self-adj. and $f(W) \subset W$.

then $f|W : W \to W$ also self adj.

By induction: Let $f: V \to V$ be self-adjoint.

Let $v \in V$ be an eigenvector of f:

$$f(v) = \lambda v.$$ by 3

$$f(\mathrm{span}(v)) \subset \mathrm{span}(v).$$

$$f(\mathrm{span}(v)^{\perp}) \subset \underbrace{\mathrm{span}(v)^{\perp}}_{\text{Euclidean VS}}$$ by 2

f is self adj. operator on $\mathrm{span}(v)^{\perp}$

$$\dim(\mathrm{span}(v)^{\perp}) < \dim V.$$

By induction, we may assume that $f\big|_{\mathrm{span}(v)^\perp}$ admits an ON basis of eigenvectors.

Then add normal. of v to this basis

get ON basis of V

consisting of eigenvectors of f.

Spectral theorem proved. $\square$