Endomorphisms of a Euclidean Vector Space

Let $V$ be a Euclidean VS. $B = (v_1, \ldots, v_n)$ an ON basis.

1. Orthogonal endomorphisms.

$\varphi : V \to V$ is orth if $\forall v, w \in V : \langle \varphi(v), \varphi(w) \rangle = \langle v, w \rangle$.

Let $A = M_B(\varphi)$.

$\varphi$ is orth. $\iff \varphi(v_1), \varphi(v_2)$ is an ON basis in $V$.

$\iff$ Columns of $A$ form an ON basis of $\mathbb{R}^n$.

$\iff A^t A = I_n$

$\iff A$ is invertible and $A^{-1} = A^t$.

$\iff A A^t = I_n$

$\iff$ Rows of $A$ form an ON basis of $\mathbb{R}^n$.

$\iff A$ is an orthogonal matrix.
Groups: the orth endomorphisms of $V$

$O(V)$

Form a group.

A group is a pair $(G, \cdot)$ set and

- $G \times G \rightarrow G$
  - $(g, h) \rightarrow g \cdot h$.

Properties:
1. assoc.
2. Funct m.G.
3. $\forall g \in G \exists$ inverse of $g$.

$O(V) \times O(V) \rightarrow O(V)$

$(g, h) \rightarrow g \cdot h$ composition.

The operation is defined b/c

If $g$ orth, $h$ orth $\Rightarrow g \cdot h$ orth.

Axioms:
- $id_V$ is orth.

If $g$ is orth, it is invertible and

- $g^{-1}$ is also orth.

$\langle g^{-1}v, g^{-1}w \rangle = \langle g(g^{-1}v), g(g^{-1}w) \rangle$

$\langle v, w \rangle$.
So \( O(V) \) is a group.

\[ V = (\mathbb{R}^n, \text{std}). \quad O(n) = O(\mathbb{R}^n, \text{std}). \]

\[ O(n) = \{ A \in M(n \times n, \mathbb{R}) \mid A^t A = I \} \]

\[ A \in O(n) \text{ then } A^t A = I_n \]

\[ \det (A^t A) = 1 \]

\[ \det (A^t) \det A = 1 \]

\[ (\det A)^2 = 1. \quad \forall \ A \in \mathbb{R} \]

\[ \text{so } \det A = \pm 1. \]

\[ \text{if } \det A = \pm 1 \text{ then } \]

\[ \text{SO}(n) = \{ A \in O(n) \mid \det A = 1 \} \]

\[ = \text{special orthogonal group}. \]

\[ O(2) = \text{SO}(2) \cup \{ A \in O(2) \mid \det A = -1 \} \]

\[ \text{rotations} \quad \text{reflections}. \]
SO(3): \( \text{let } A \in SO(3). \)

Claim: \( A \) has a non-trivial fixed vector \( v \neq 0 \):
\[ Av = v. \]

Equivalently:
Claim: \( 1 \) is an eigenvalue of \( A \).

General fact: \( A \in O(n) \) then and \( \lambda \in \mathbb{R} \) is an
eigenvalue of \( A \) then \( \lambda = \pm 1. \)

Proof. \[ Av = \lambda v \quad v \neq 0 \]
\[ \langle Av, Av \rangle = \langle \lambda v, \lambda v \rangle \]
\[ \langle v, v \rangle = \lambda^2 \langle v, v \rangle \]
\[ \|v\|^2 = \lambda^2 \|v\|^2 \]
\[ (1-\lambda^2) \|v\|^2 = 0. \]
\[ \left\{ \begin{array}{l} \neq 0 \\
\end{array} \right. \]
\[ \Rightarrow 1 - \lambda^2 = 0 \]
\[ \lambda = \pm 1. \]
Char poly of $A \in SO(3)$:

$\det (tI_3 - A)$

$\det (A - tI_3)$

degree 3 poly.

constant term is $(t=0)$ $\det A = 1$.

graph of a cubic with leading term $-t^3$

graph of $\det (A + tI_7)$

by the intermediate value theorem, char poly has a positive root.

so $A$ has a positive EV.

so $A$ has 1 as EV.
So let \( u \) be a unit eigenvector corr.

do to \( \lambda = 1 \).

Then complete to an ON basis of \( \mathbb{R}^3 \):

\[
\mathbf{B} = (u_1, u_2, u_3) \text{ ON basis}
\]

\[
[A]_{\mathbf{B}} \text{ is an orth. matrix (w/ } A \text{ orth, } \mathbf{B} \text{ ON.})
\]

\[
[A]_{\mathbf{B}} = \begin{pmatrix}
1 & a_1 & b_1 \\
0 & a_2 & b_2 \\
0 & a_3 & b_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} = 0 \Rightarrow a_1 = 0
\]

\[
b_1 = 0
\]

\[
[A]_{\mathbf{B}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & a_2 & b_2 \\
0 & a_3 & b_3
\end{pmatrix}
\]

2x2 orth matrix w/ \( \det 1 \).

So a rotation matrix.

\[
[A]_{\mathbf{B}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{pmatrix}
\]

So ON basis in which the orth trans.

Looks like this
In 3D coords, the matrix $A = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ describes the rotation about the $\theta$-axis by angle $\alpha$.

So conclude: $A$ is rotation by angle $\alpha$ about span $(u_1) = \text{line spanned by a fixed vector}$.

So $SO(3) = \{ \text{rotations about axes through origin in } \mathbb{R}^3 \}$. 
2. **Self-adjoint endomorphisms.**

\[ q : V \to V \text{ self-adjoint } \iff \forall v, w \in V : \langle v, qw \rangle = \langle qv, w \rangle. \]

**Remark.** If \( q : V \to V \) endomorphism \((V, \text{Euclidean})\), \( B \) on basis of \( V \) then

\[ A = [q]_B \]

then columns \( A_j = [q(V_j)]_B = \begin{pmatrix} \langle v_1, q(v_j) \rangle \\ \vdots \\ \langle v_n, q(v_j) \rangle \end{pmatrix} \)

the \( i \)th entry of \( j \)th col is \( \langle v_i, q(v_j) \rangle \)

\[ a_{ij} = \langle v_i, q(v_j) \rangle \]

\( q \) self-adjoint \( \iff \forall i, j : \langle v_i, q(v_j) \rangle = \langle q(v_i), v_j \rangle \)

(Enough to check the self-adjoint property on basis vectors.)

\[ \iff \forall i, j : a_{ij} = \overline{a_{ji}} \]

\[ \iff A = \overline{A^T} \ (A \text{ symmetric}) \]
Theorem (spectral theorem!) Let $V$ be Euclidean, $\dim V < \infty$.

Let $f : V \rightarrow V$ be a self-adjoint operator. Then

1. $f$ is diagonalizable.

2. If $\lambda, \mu \in \mathbb{R}$ are eigenvalues of $f$ then $E_\lambda \perp E_\mu$.

(In particular there exists an ON basis consisting of eigenvectors for $f$.)

(take ON basis for all $E_\lambda$, put them together.)

In particular $V = \mathbb{R}^n$, std. $f = A$.

For every symmetric matrix $A$, there exists an orth matrix $P$ s.t.

\[
A = PDP^{-1} \quad \text{D: diagonal.}
\]

\[
A = PD P^T \quad P \text{ orth.}
\]

Proof: $A \in M(n \times n, \mathbb{R}) \subset M(n \times n, \mathbb{C})$. 

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