# Endomorphisms of a Euclidean Vector Space



Let $V$ be a Euclidean VS. $\mathcal{B}=(v_1,\dots,v_n)$ an ON basis.

① orthogonal endomorphisms.

$\varphi : V \to V$ is orth if $\forall v,w \in V$: $\boxed{\langle \varphi v, \varphi w\rangle = \langle v,w\rangle}$.

Let $A = M_{\mathcal{B}}(\varphi)$.

$$\begin{tikzcd} V \arrow[r, "\varphi"] & V \\ \mathbb{R}^n \arrow[u, "\Phi_{\mathcal{B}}"] \arrow[r, "A"] & \mathbb{R}^n \arrow[u, "\Phi_{\mathcal{B}}"'] \end{tikzcd}$$

$\Phi_{\mathcal{B}} \leftarrow$ orth.

$\varphi$ is orth. $\iff$ $\varphi(v_1),\dots,\varphi(v_n)$ is an ON basis in $V$.

$\iff$ Columns of $A$ form an ON basis of $\mathbb{R}^n$.

$\iff$ $A^t A = I_n$

$\iff$ $A$ is invertible and $A^{-1} = A^t$.

$\iff$ $A A^t = I_n$

$AA^{-1} = I$

$A^{-1}A = I$

$\iff$ Rows of $A$ form an ON basis of $\mathbb{R}^n$.

$\iff$: $A$ is an orthogonal matrix.

Groups: the orth endomorphisms of $V$

$O(V)$

form a group.

A group is a pair $(G, \cdot)$ set and

$\cdot \ G \times G \longrightarrow G$
$(g, h) \longmapsto g \cdot h$.

properties: (1) $\cdot$ assoc.
(2) $\exists$ unit in $G$.
(3) $\forall g \in G \ \exists$ inverse of $G$.

$$O(V) \times O(V) \longrightarrow O(V)$$
$$(\varphi, \psi) \longmapsto \varphi \circ \psi \quad \text{composition.}$$

the operation is defined b/c
$\varphi$ orth, $\psi$ orth $\Rightarrow$ $\varphi \circ \psi$ orth.

Axioms: $id_V$ is orth.
if $\varphi$ is orth, it is invertible and
$\varphi^{-1}$ is also orth:

$$\langle \varphi^{-1} v, \varphi^{-1} w \rangle \underset{\varphi \text{ orth}}{=} \langle \varphi(\varphi^{-1} v), \varphi(\varphi^{-1} w) \rangle$$
$$= \langle v, w \rangle.$$

So $O(V)$ is a group.

$V = (\mathbb{R}^n, \text{std})$. $\quad O(n) = O(\mathbb{R}^n, \text{std})$.

$$O(n) = \{ A \in M(n\times n, \mathbb{R}) \mid A^t A = I_n \}.$$

~~Ex 1 O(2) =~~

$A \in O(n)$ then $\quad A^t A = I_n$

$$\det(A^t A) = 1$$

$$\det(A^t) \det A = 1$$

$$(\det A)^2 = 1. \quad \text{in } \mathbb{R}$$

so $\det A = \pm 1$.

if ~~det A = ±1 then~~

$$SO(n) = \{ A \in O(n) \mid \det A = 1 \}.$$

$=$ <u>special orthogonal group</u>.

$$O(2) = \underbrace{SO(2)}_{\text{rotations}} \sqcup \underbrace{\{ A \in O(2) \mid \det A = -1 \}}_{\text{reflections.}}$$

**SO(3):** Let $A \in SO(3)$.

**Claim:** $A$ has a non-trivial fixed vector $v \neq 0$:

$Av = v$.

Equivalently: **Claim:** $1$ is an eigenvalue of $A$.

**General fact:** $A \in O(n)$ ~~then~~ and $\lambda \in \mathbb{R}$ is an eigenvalue of $A$ then $\lambda = \pm 1$.

Proof.

$$Av = \lambda v \qquad v \neq 0$$

$$\langle Av, Av \rangle = \langle \lambda v, \lambda v \rangle$$

$$\langle v, v \rangle = \lambda^2 \langle v, v \rangle$$

$$\|v\|^2 = \lambda^2 \|v\|^2$$

$$(1-\lambda^2) \underbrace{\|v\|^2}_{\neq 0} = 0.$$

$$\Rightarrow \quad 1-\lambda^2 = 0$$

$$\lambda = \pm 1. \quad \square$$

Char poly of $A \in \underline{SO(3)}$:

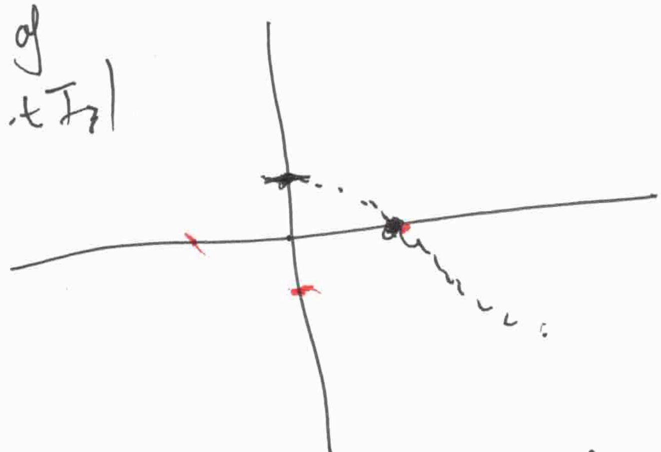
$\cancel{\det(tI_3 - A)}$

$\det(A - tI_3)$  degree 3 poly. $\begin{vmatrix} a_{11}-t & \cdot & \cdot \\ \cdot & a_{22}-t & \cdot \\ \cdot & \cdot & a_{33}-t \end{vmatrix}$

constant term is $(t=0)$ $\det A = 1$.

graph of a cubic with leading term $\boxed{-t^3}$ $\underline{\text{const. term} = 1}$.

graph of $\det(A - tI_3)$



by the intermediate value theorem, char poly has a positive root.

So $A$ has a positive EV.

So $A$ has $1$ as EV.

cubic w/ leading term $-t^3$

So let $u_1$ be a unit ~~vector~~ eigenvector

corr. to EV 1.

then complete to an ON basis of $\mathbb{R}^3$:

$\mathcal{B} = (u_1, u_2, u_3)$ ON basis.

$[A]_{\mathcal{B}}$ is an orth. matrix (b/c A orth, $\mathcal{B}$ is ON)

$$[A]_{\mathcal{B}} = \underbrace{\begin{pmatrix} 1 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a_3 & b_3 \end{pmatrix}}. \quad = \begin{pmatrix} 1 & 1 & 1 \\ 0 & a & b \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \Rightarrow a_1 = 0$$

$$b_1 = 0$$

$$[A]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & b_2 \\ 0 & a_3 & b_3 \end{pmatrix}$$

2x2 orth matrix w/ det 1.

so a rotation matrix.

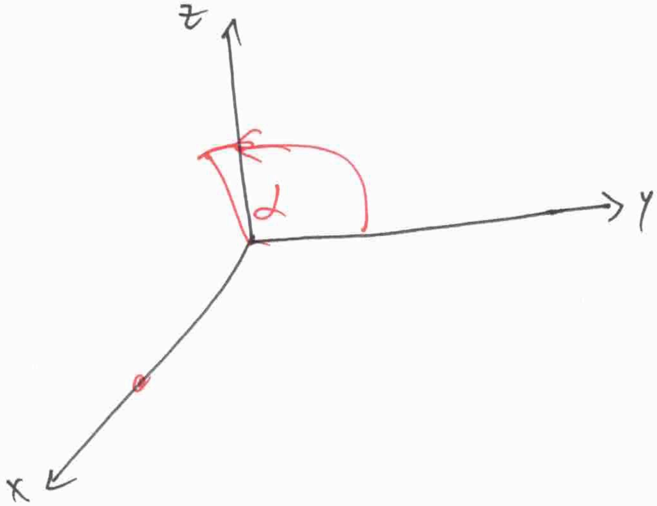
$$\boxed{[A]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \quad \alpha \in \mathbb{R}.}$$

So $\exists$ ON basis in which the orth transf. looks like this

In std coords, the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ \\ \end{pmatrix}$$

describes the rotation about the ~~z~~ x-axis by angle $\alpha$.




$(1\ 0\ 0$

So conclude: $A$ is rotation by angle $\alpha$ about $\mathrm{span}(u_1)$ = line spanned by a fixed vector.


So $SO(3) = \{$rotations about axes through origin in $\mathbb{R}^3\}$.

## (2) self-adjoint endomorphisms.

$\varphi : V \to V$ <u>self-adjoint</u> if

$\forall v, w \in V : \quad \langle v, \varphi w \rangle = \langle \varphi v, w \rangle$.

<u>Rmk.</u> If $\varphi : V \to V$ endomorphism ($V$ Euclidean).
$\mathcal{B}$ ON basis of $V$ ~~then~~

$$A = [\varphi]_{\mathcal{B}}$$

then columns $\quad A_j = [\varphi(v_j)]_{\mathcal{B}} = \begin{pmatrix} \langle v_1, \varphi(v_j) \rangle \\ \vdots \\ \langle v_n, \varphi(v_j) \rangle \end{pmatrix}$

the $i^{th}$ entry of $j^{th}$ col is $\langle v_i, \varphi(v_j) \rangle$

$$\boxed{a_{ij} = \langle v_i, \varphi(v_j) \rangle}$$

$\varphi$ self adjoint $\iff \forall i,j : \quad \underbrace{\langle v_i, \varphi v_j \rangle}_{} = \langle \varphi v_i, v_j \rangle \overset{\langle v_j, \varphi v_i \rangle}{}$

(enough to check the self-adj property on basis vectors.)

$\iff \forall i,j : \quad a_{ij} = a_{ji}$

$\iff \quad A = A^t \quad$ ($A$ symmetric)

Theorem (spectral theorem!) Let $V$ be Euclidean, $\dim V < \infty$

Let $f : V \longrightarrow V$ be a self-adjoint operator. Then

1. $f$ is diagonalizable!

2. If $\lambda \neq \mu \in \mathbb{R}$ are eigenvalues of $f$ then
$E_\lambda \perp E_\mu$.

(In particular there exists an ON basis consisting of eigenvectors for $f$.)

( take ON basis for all $E_\lambda$ , put them together..)

In particular $V = \mathbb{R}^n$, std. $f = A$,

For every symmetric matrix $A$ there exists an orth matrix $P$ s.t.

$$\boxed{\begin{array}{ll} A = PDP^{-1} & D \text{: diagonal.} \\ A = PDP^{t} & P \text{ orth.} \end{array}}$$

Proof: $A \in M(n \times n, \mathbb{R}) \subset M(n \times n, \mathbb{C})$.

Dvetkis