Euclidean Vector Spaces III

Matrices and Bilinear maps.

Let V, W be two finite-dim'l VS/$\mathbb{F}$.

Let $\mathcal{B} = (v_1, \ldots, v_n)$ basis of V

$\mathcal{C} = (w_1, \ldots, w_m)$ basis of W.

Let $\beta : V \times W \longrightarrow \mathbb{F}$ be bilinear.

be a bilinear form (base field).

β is linear in each of the 2 arguments.

$\forall v \in V \ \forall w \in W: \quad \beta(v, w) \in \mathbb{F}$.

$\beta(v, \cdot) : W \to \mathbb{F}$ linear $\forall v \in V$

$\beta(\cdot, w) : V \to \mathbb{F}$ linear. $\forall w \in W$.

the matrix of β w.r.t. $\mathcal{B}, \mathcal{C}$ call it A:

$A = (a_{ij})$ is defined to be $a_{ij} = \beta(v_i, w_j)$ $\quad$ i = row index $\quad n$ rows

j = col. index $\quad m$ cols

$$A \in M(n \times m, \mathbb{F}).$$

Given the matrix A of the bilinear form β,

we know β entirely:

$v \in V$ arbitrary $\qquad v = \sum_i \lambda_i v_i \qquad \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = [v]_{\mathcal{B}}$

$w \in W$ -"- . $\qquad w = \sum_j \mu_j w_j \qquad \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} = [w]_{\mathcal{C}}$.

coordinate vectors.

then $\beta(v,w) = \beta\left(\sum_i \lambda_i v_i, \sum_j \mu_j w_j\right)$

$= \sum_i \lambda_i \, \beta\left(v_i, \sum_j \mu_j w_j\right)$ linearity of β in 1st arg.

$= \sum_i \lambda_i \sum_j \mu_j \, \beta(v_i, w_j)$ — " — 2nd.

$= \sum_{ij} \lambda_i \, \beta(v_i, w_j) \, \mu_j$

$= \sum_{ij} \lambda_i \, a_{ij} \, \mu_j$ product of 3 matrices

$\sum_{ij} \lambda_{i1} a_{ij} \mu_{j1}$

$$= (\lambda_1 \ldots \lambda_n) \; A \; \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}$$

$$\underbrace{1 \times n \quad n \times m \quad m \times 1}_{1 \times 1}$$

$$\boxed{\beta(v,w) = [v]_{\mathcal{B}}^t \, A \, [w]_{\mathcal{C}}} \quad (*)$$

expresses β in terms of its matrix A and the coordinate vectors of v, w.

So β is determined by its matrix

$$A = {}_{\mathcal{B}}M_{\mathcal{E}}(\beta) \leftarrow$$ notation for the matrix. depends on β, $\mathcal{B}$, $\mathcal{E}$.

$$\boxed{\beta(v,w) = [v]_{\mathcal{B}}^{t} \; {}_{\mathcal{B}}M_{\mathcal{E}}(\beta) \; [w]_{\mathcal{E}}}$$

Conversely, Every ~~A~~ $A \in M(n \times m, \mathbb{F})$

defines a bilinear form $\beta : V \times W \to \mathbb{F}$

by the formula (*).

there is a certain analogy:

linear maps have matrices	bilinear forms have matrices
$V \xrightarrow{f} W$	$V \times W \xrightarrow{\beta} \mathbb{F}$
if you choose bases for V,W	if you choose bases for V,W.

important case is $V \times V \xrightarrow{\beta} \mathbb{F}$

bilinear form on V. (Same vector space.)

then we take the same basis

and write $M_{\mathcal{B}}(\beta) = {}_{\mathcal{B}}M_{\mathcal{B}}(\beta)$.

$$\begin{tikzcd} (\Phi_B x, \Phi_{\mathcal{C}}(y)) & V \times W \arrow[r, "\beta"] & \mathbb{F} \\ (x,y) \arrow[u, mapsto] & \mathbb{F}^n \times \mathbb{F}^m \arrow[u, "\Phi_B \times \Phi_{\mathcal{C}}"] \arrow[r, "A"] & \mathbb{F} \arrow[u, "\text{identity}"'] \end{tikzcd}$$

diagram of maps

. (§) diagram commutes.

$$\left(\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix}, \begin{pmatrix}\mu_1\\ \vdots\\ \mu_m\end{pmatrix}\right) \longmapsto (\lambda_1 \ldots \lambda_n) A \begin{pmatrix}\mu_1\\ \vdots\\ \mu_m\end{pmatrix}$$

$\longleftarrow$ For every matrix $A \in M(n\times m, \mathbb{F})$ this formula defines a bilinear form $\mathbb{F}^n \times \mathbb{F}^m \longrightarrow \mathbb{F}$.

$$\begin{tikzcd} (v, w) \arrow[r, mapsto] & \beta(v,w) \arrow[d, equal] & \text{see } (*) \\ ([v]_B, [w]_{\mathcal{C}}) \arrow[u, mapsto] \arrow[r, mapsto] & {[v]_B^t A [w]_{\mathcal{C}}} & \end{tikzcd}$$

what the maps do on elements

Inner products are very special bilinear forms:

inner product:
- bilinear. $V \times W \longrightarrow \mathbb{F}$
- symmetric: $V \times V \longrightarrow \mathbb{F}$ need $V = W$ for symmetric to make sense.
- positive def. $V \times V \longrightarrow \mathbb{R}$ need $\mathbb{F} = \mathbb{R}$ for pos. def. to make sense.

Gram Schmidt: Given inner product on V. $\langle , \rangle : V \times V \longrightarrow \mathbb{R}$. $\boxed{\beta = \langle , \rangle}$

($\dim V < \infty$) then there exists an ON basis of V. $\mathcal{B} = (v_1, \ldots, v_n)$ ON basis.

$\mathcal{B}$ ON basis for $\langle , \rangle$

$$\Longleftrightarrow$$

$$\langle v_i, v_j \rangle = \left\{ \begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{array} \right\} =: \delta_{ij} \quad \text{"Kronecker } \delta\text{"}$$

$$\Longleftrightarrow$$

$$M_{\mathcal{B}}(\langle , \rangle) = (\delta_{ij}) = I_n$$

Special case of diagram (§) for $\beta = \langle , \rangle$ an $\mathcal{B}$ an ON basis.

$$\begin{tikzcd} V \times V \arrow[r, "\langle , \rangle"] & \mathbb{R} \\ \mathbb{R}^n \times \mathbb{R}^n \arrow[u, "\Phi_{\mathcal{B}} \times \Phi_{\mathcal{B}}"] \arrow[r, "\cancel{\cdot}"', "\text{std. inner product}"'] & \mathbb{R} \arrow[u, "id"'] \end{tikzcd}$$

if $\mathcal{B}$ is an ON basis for $\langle , \rangle$

$v, w \in \mathbb{R}^n$:

$$v \cdot w = {}^t v\, w = \langle \Phi(v), \Phi(w) \rangle$$

the statement "Every inner product admits an ON basis:" can be expressed as "Every Euclidean vector space is $\underline{\underline{\text{isometric}}}$ to $(\mathbb{R}^n, \text{std})$".

the precise meaning of isometric comes next:

Orthogonal Maps (see Jänich)

Definition. Let V, V' be Euclidean vector spaces. A linear map

$$V \xrightarrow{f} V'$$

is <u>orthogonal</u> or <u>an isometry</u> if

$\forall\, v, w \in V$:

$$\langle v, w \rangle = \langle f(v), f(w) \rangle.$$

inner product in V ... inner product in V'.

Fact. Every orthogonal map is injective!

Pf. Assume $f: V \to V'$ is orthogonal.

Let $v \in V$ and $f(v) = 0$. to prove injective, prove $\ker = \{0\}$.

then $\langle v, v \rangle = \langle f(v), f(v) \rangle = \langle 0, 0 \rangle = 0$.

$\Rightarrow v = 0$ (by pos. def. of $\langle , \rangle$ on V.) $\square$

Cor. $\dim V < \infty$ and $f: V \to V$ is an orth. endomorphism, then f is an isomorphism.

<u>Example.</u> Let $(V, \langle , \rangle)$ be a Euclidean VS

Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ON basis of V.

and
$$\begin{aligned} \Phi : \mathbb{R}^n &\longrightarrow V \\ e_i &\longmapsto v_i \end{aligned}$$
the can. basis isom.

Then
$$\begin{array}{ccc} \Phi : \mathbb{R}^n & \longrightarrow & V \\ \uparrow & & \uparrow \\ \text{std.} & & \langle , \rangle \end{array}$$
is an orthogonal isomorphism.
(an isometry).

b/c $\langle \Phi(v), \Phi(w) \rangle = v \cdot w$,

(making the above use of the word "isometric" precise.)

<u>Fact:</u> Let V, V' be Euclidean VS.

$(v_1 .. v_n)$ an ON basis of V

then a linear map $f: V \longrightarrow V'$ is orthogonal

$\Longleftrightarrow$

$\big(f(v_1), .., f(v_n)\big)$ is an ON system in V'.

Pf see Jänich.

<u>Definition</u> $\boxed{O(V) = \{ f: V \to V \mid f \text{ orthogonal} \}}$.

if $V = (\mathbb{R}^n, \text{std})$

write $O(n) = O(V)$.

$=$ <u>orthogonal matrices.</u>

Via std basis of $\mathbb{R}^n$ endomorphisms of $\mathbb{R}^n$ are matrices.

$A \in M(n \times n, \mathbb{R})$ is <u>orthogonal</u> if and only if.

$A(e_1), .., A(e_n)$ form an ON system.

Cols of A form an ON system.

$A_1 .. A_n$ cols of A.

$A_i \cdot A_j = \delta_{ij}$

$$\iff A_i^t A_j = \delta_{ij} \quad \forall ij$$

$$\iff \boxed{A^t A = \mathrm{Id}} \quad \begin{pmatrix} - & A_1^t & - \\ & \vdots & \\ - & A_n^t & - \end{pmatrix} \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix}$$

$$O(n) = \{ A \in M(n \times n, \mathbb{R}) \mid A^t A = \mathrm{id} \}.$$

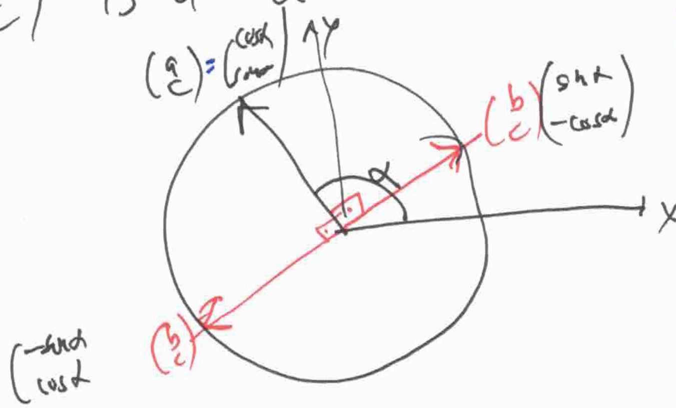
<u>Example</u>: Find all matrices in $O(2)$:

O(2):

orth. transf. $\mathbb{R}^2 \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \mathbb{R}^2$ . .

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is orth. $\iff$ $\begin{pmatrix} a \\ c \end{pmatrix}$ $\begin{pmatrix} b \\ d \end{pmatrix}$ is an ON basis of $\mathbb{R}^2$.

$\begin{pmatrix} a \\ c \end{pmatrix}$ is a unit vector: so $\exists\, \alpha \in \mathbb{R}$: such that $\boxed{\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}}$.



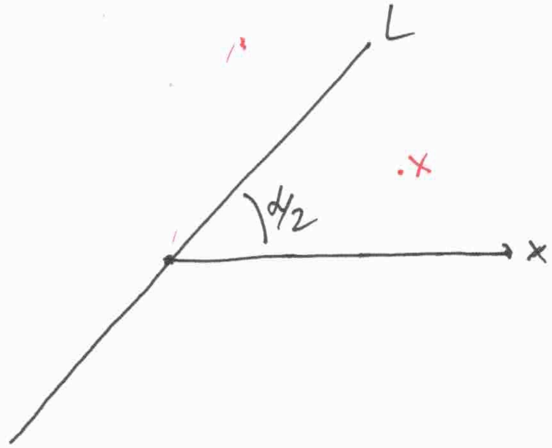
Then $\begin{pmatrix} b \\ d \end{pmatrix} \perp \begin{pmatrix} a \\ c \end{pmatrix}$ so there are two choices for $\begin{pmatrix} b \\ d \end{pmatrix}$:

by some simple trigonometry: either $\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$

or $\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} -\cos\alpha \\ \sin\alpha \end{pmatrix}$.

Case 1: $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ rotation by angle α

Case 2: $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ reflection across line at $\alpha/2$



So all isometries $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are either rotations about the origin or reflections across a line through the origin.