Euclidean Vector Spaces II

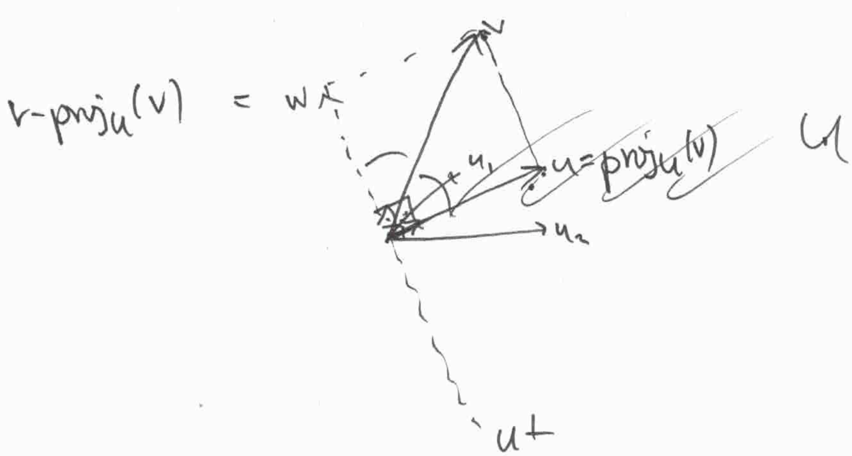
Let V be a Euclidean vector space.

And let $u_1, \dots, u_n$ be an ON system in V.

Then let $U = \mathrm{span}(u_1, \dots, u_n) \subset V$ subspace.

Claim: Then every vector $v \in V$ can be written uniquely as

$$v = u + w \qquad u \in U,\ w \in U^\perp.$$



Moreover: Formula for u:

$$u = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$$

then $w = v - u = v - \sum_{i=1}^{n} \langle v, u_i \rangle u_i \ \in U^\perp.$

Pf: to prove $w \perp U$ suffices $u \perp u_i$ for $i = 1, \dots, n$.

Then $\langle w, u_i \rangle = \langle v - \sum_{j=1}^{n} \langle v, u_j \rangle u_j , u_i \rangle$

$$= \langle v, u_i \rangle - \sum_{j=1}^{n} \langle v, u_i \rangle \underbrace{\langle u_j, u_i \rangle}_{=\begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}$$

$$= \langle v, u_i \rangle - \langle v, u_i \rangle 1$$

$$= 0. \quad \square.$$

We have proved it is possible to write v in this way. <u>Uniqueness</u>:

$$v = u + w = u' + w' \qquad u, u' \in U$$
$$w, w' \in U^\perp.$$

then $\underbrace{u - u'}_{\in U} = \underbrace{w' - w}_{U^\perp}$.

but $U \cap U^\perp = \{0\}$ because:

if $v \in U \cap U^\perp$ then $v \perp v$ so $\langle v, v \rangle = 0$

so $\|v\|^2 = 0$

so $v = 0$

by positive definitenes of $\langle , \rangle$.

So $u - u' = 0 = w - w'$

so $u = u'$, $w = w'$. (Claim on page 1 proved.)

for $v \in V$ we call

$$\underbrace{\mathrm{proj}_{\mathcal{U}}(v)}_{\mathcal{U}=} = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$$

the <u>orthogonal projection</u> of v onto $\mathrm{Span}(u_1, \dots, u_n)$

$u_1, \dots, u_n$ ON system.

<u>Remark:</u> $\measuredangle\left(v, \mathrm{proj}_{\mathcal{U}}(v)\right) \leq \frac{\pi}{2}$

and $\measuredangle\left(v, v - \mathrm{proj}_{\mathcal{U}}(v)\right) \leq \frac{\pi}{2}$

And w = the orth projection onto $\mathrm{Span}(u_1, \dots, u_n)^{\perp}$

equivalently $\cos\measuredangle \geq 0$

— " — $\langle v, \mathrm{proj}_{\mathcal{U}}(v) \rangle \geq 0$

$\langle v, v - \mathrm{proj}(v) \rangle \geq 0.$

See Claim on page -4-

$$v = \underline{v - \mathrm{proj}_{\mathcal{U}}(v)} + \underline{\mathrm{proj}_{\mathcal{U}}(v)}$$

$$\langle v, v \rangle = \langle v, v - \mathrm{proj} \rangle + \langle v, \mathrm{proj} \rangle$$

$$\|v\|^2 = \langle v, v - \mathrm{proj} \rangle + \langle v, \sum_{i=1}^{n} \langle v, u_i \rangle u_i \rangle$$

$$= \langle v, v - \mathrm{proj} \rangle + \sum_{i=1}^{n} \langle v, u_i \rangle \langle v, u_i \rangle$$

$$\|v\|^2 = \underbrace{\langle v, v - \mathrm{proj} \rangle}_{\geq 0\,??} + \underbrace{\sum_{i=1}^{n} \langle v, u_i \rangle^2}_{\geq 0}$$

$\Rightarrow \langle v, \mathrm{proj} \rangle \geq 0.$

~~$\langle v, v \rangle = \langle v, v - \text{proj} \rangle + \langle v, \text{proj} \rangle$~~

$\langle v, v \rangle = \langle v, v-u \rangle + \langle v, u \rangle$

$$\sum_{i=1}^{r} \langle v, u_i \rangle^2 \leq \|v\|^2$$

this will follow from Cauchy-Schwarz.

<u>ex:</u> $n=1$ then $\text{proj}_U(v) = \langle v, u_1 \rangle u_1$.

then <u>Claim</u> $\langle v, u_1 \rangle^2 \leq \|v\|^2$ u_1 unit.

$\langle v, u_1 \rangle^2 \leq \|v\|^2 \|u_1\|^2$ C-S.

$v \in V$ $u = \text{proj}_U(v)$ $U = \text{span}(u_1, \dots, u_n)$ $u_1, \dots, u_n$ ON system.

(*) $v = (v-u) + u$. <u>Claim:</u> $\langle v, v-u \rangle \geq 0$. and $\langle v, u \rangle \geq 0$

<u>Proof:</u> Since $u \perp (v-u)$ by Pythagoras (see Jänich / Homework)

$\|v\|^2 = \|v-u\|^2 + \|u\|^2$ in particular, $\|u\|^2 \leq \|v\|^2$.

Then $\langle u, u \rangle \leq \langle v, v \rangle$.

We have $\langle v, v \rangle = \langle v, v-u \rangle + \langle v, u \rangle$ apply $\langle v, \cdot \rangle$ to (*)

and $\langle u, v \rangle = \underbrace{\langle u, v-u \rangle}_{=0} + \underbrace{\langle u, u \rangle}_{\geq 0}$ apply $\langle u, \cdot \rangle$ to (*)

so $\langle v, v \rangle = \langle v, v-u \rangle + \langle u, u \rangle$

or $\|v\|^2 = \langle v, v-u \rangle + \|u\|^2$

so $\langle v, v-u \rangle \geq 0$, claim proved. $\square$

Gram-Schmidt Orthonormalization.

How to construct orthonormal bases?

Start with a linearly independent family $v_1, \ldots, v_n$ in V.

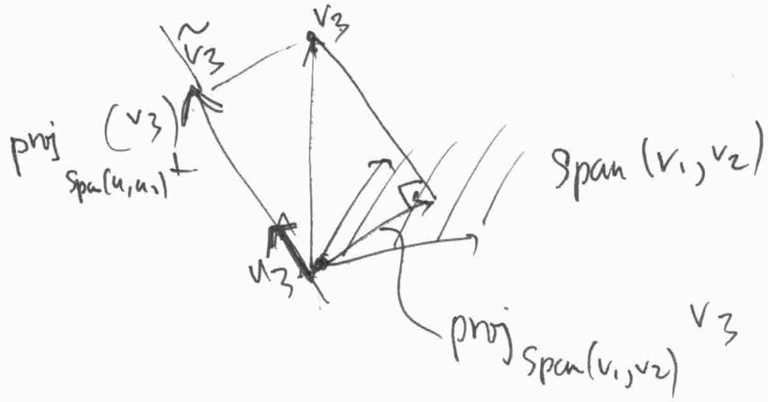
Convert it into an ON system $u_1, \ldots, u_n$

s.t. $\operatorname{span}(u_1, \ldots, u_n) = \operatorname{span}(v_1, \ldots, v_n)$.

Define $\underline{\forall k = 0, 1, \ldots, n-1}$ u_{k+1} to be the normalization

of $\tilde{v}_{k+1}$ = projection of v_{k+1} onto the orthogonal complement of $\operatorname{span}(v_1, \ldots, v_k)$.

e.g. u_1 = norm of $\tilde{v}_1 = v_1$ $\qquad u_1 = \dfrac{v_1}{\|v_1\|}$.



① the normalization is defined:

Since $v_{k+1} \notin \text{span}(v_1, \dots, v_k)$.

the proj onto $\text{span}(v_1, \dots, v_k)^{\perp}$ is non-zero.

② Also, $\tilde{v}_{k+1} \perp \text{span}(v_1, \dots, v_k)$.

So $u_{k+1} \perp \text{span}(v_1, \dots, v_k)$.

③ $\text{span}(u_1, \dots, u_{k+1}) = \text{span}(v_1, \dots, v_{k+1})$:

by induction: assume $\boxed{\text{span}(v_1, \dots, v_k) = \text{span}(u_1, \dots, u_k)}$

$$v_{k+1} = \underbrace{\left(v - \text{proj}_{\text{span}(v_1 \dots v_k)} v_{k+1}\right)}_{\tilde{v}_{k+1}} + \text{proj}_{\text{span}(v_1 \dots v_k)} v_{k+1}$$

$$= \text{proj}_{\text{span}(v_1 \dots v_k)^{\perp}}(v_{k+1})$$

we see that $v_{k+1} \in \text{span}(v_1, \dots, v_k, \tilde{v}_{k+1})$

$= \text{span}(u_1, \dots, u_k, u_{k+1})$

$\Rightarrow \underline{\text{span}(v_1, \dots, v_{k+1}) \subset \text{span}(u_1, \dots, u_{k+1})}$.

Also $\tilde{v}_{k+1} \in \text{span}(v_1 \dots v_k, v_{k+1})$

$\Rightarrow u_{k+1} \in \text{span}(v_1 \dots v_{k+1}) \Rightarrow \text{span}(u_1 \dots u_{k+1}) \subset \text{span}(v_1 \dots v_{k+1})$.

Then since u_{k+1} is unit

and $u_{k+1} \perp \mathrm{span}(v_1 \ldots v_k)$

$= \mathrm{span}(u_1 \ldots u_k)$.

By induction $u_1 \ldots u_k$ is an ON system

so also $u_1 \ldots u_{k+1}$ is an ON system.

So $u_1 \ldots u_n$ is an ON system.

Formulas: recursive:

$$u_{k+1} = \frac{\tilde{v}_{k+1}}{\|\tilde{v}_{k+1}\|} \quad \text{and} \quad \boxed{\tilde{v}_{k+1} = v_{k+1} - \underbrace{\sum_{i=1}^{k} \langle v, u_i \rangle u_i}_{\mathrm{proj}_{\mathrm{span}(v_1 \ldots v_k)} v_{k+1}}}.$$

QR-factorization.

$V = \mathbb{R}^m$ std. inner product.

$v_1, \dots, v_n$ lin. indep.

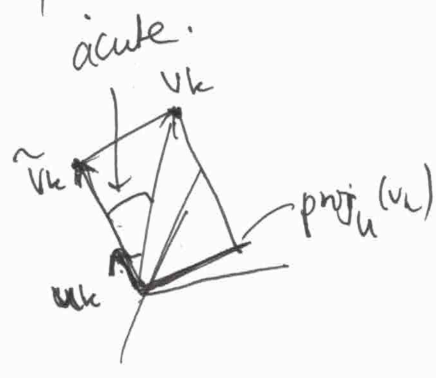
$A = (v_1 \dots v_n)$ columns $\quad A \in M(m \times n, \mathbb{R})$. $\operatorname{rank} A = n$ (so $m \geq n$).

$Q = (u_1, \dots, u_n)$ $\quad Q \in M(m \times n, \mathbb{R})$

orthonormal columns.

$$\underset{m\times n}{A} = \underset{m \times n}{Q}\ \underset{n \times n}{R} \qquad \text{where } R \in M(n \times n, \mathbb{R})$$

$$(v_1 \dots v_k \dots v_n) = (u_1 \dots u_k \dots u_n) \begin{array}{c} k^{\text{th}} \text{ colum.} \\ \begin{pmatrix} * & \dots & & * \\ & \ddots & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix} \end{array}$$



R is upper triangular b/c $\forall k$: $\quad v_k \in \operatorname{span}(u_1, \dots, u_k)$.

Also the diagonal entries of R are positive.

$r_{k,k} = \langle v_k, u_k \rangle$

$\tilde{v}_k = r_{kk} u_k$

$v_k = \tilde{v}_k + \operatorname{span}(u_1 \dots u_{k-1})$

$v_k = r_{k,k} u_k + \dots$

$r_{kk} > 0$ b/c $\measuredangle(v_k, u_k) \leq \frac{\pi}{2}$.

<u>Fact:</u> For any ~~Or~~. $A \in M(m \times n, \mathbb{R})$ of rank n.

$\exists$ $\boxed{\text{unique}}$ matrices Q, R s.t.

$$A = Q R$$

$Q \in M(m \times n, \mathbb{R})$ with ON cols.
$R \in M(n \times n, \mathbb{R})$ upper triangle w/ pos diagonal.

<u>Cor.</u> V is Euclidean.
$U \subset V$ finite-dim'l subspace.
Then U has an ON basis.

<u>Cor.</u> V Euclidean, $U \subset V$ finite dim'l.
then $V = U \oplus U^{\perp}$.

$V = U + U^{\perp}$

<u>Next:</u> orthogonal transformations.