Euclidean Vector Spaces

Length, angles.

Motivation: $\mathbb{R}^2$ $\quad$ $\mathbb{R}$-vector space.

dot product: Standard inner product on $\mathbb{R}^2$.

$$\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(v,w) \longmapsto v\cdot w = \langle v,w\rangle = v_1w_1 + v_2w_2 .$$

properties: ① dot is bilinear:

fix v then $\quad \begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ x &\longmapsto v\cdot x \end{aligned}$ $\quad$ is linear.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto v_1x_1 + v_2x_2 .$$

~~Similar~~ fix w then $\quad \begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ x &\longmapsto x\cdot w \end{aligned}$ $\quad$ is linear.

dot product is linear in each argument.
bilinear.

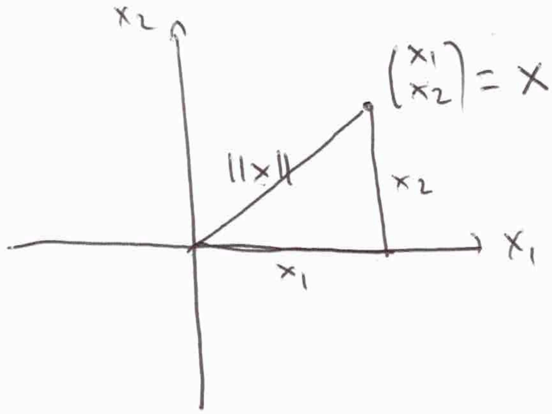
compare w/ det: $\quad \begin{aligned} \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (v,w) &\longmapsto \det(v\ w) \end{aligned}$

is bilinear.

$$\det(v\ w) = -\det(w\ v) .$$

$v\cdot w = w\cdot v$. ← symmetric

length of a vector:



Pythagoras:

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

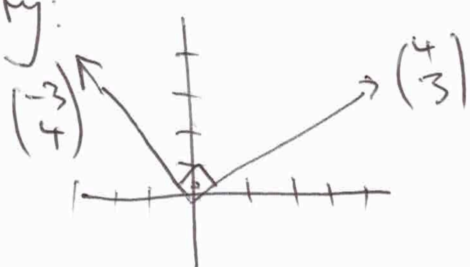
$$= \sqrt{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

$$= \sqrt{x \cdot x}$$

$$= \sqrt{\langle x, x \rangle}.$$

length of x.

Orthogonality:



$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \end{pmatrix} = -12 + 12 = 0.$$

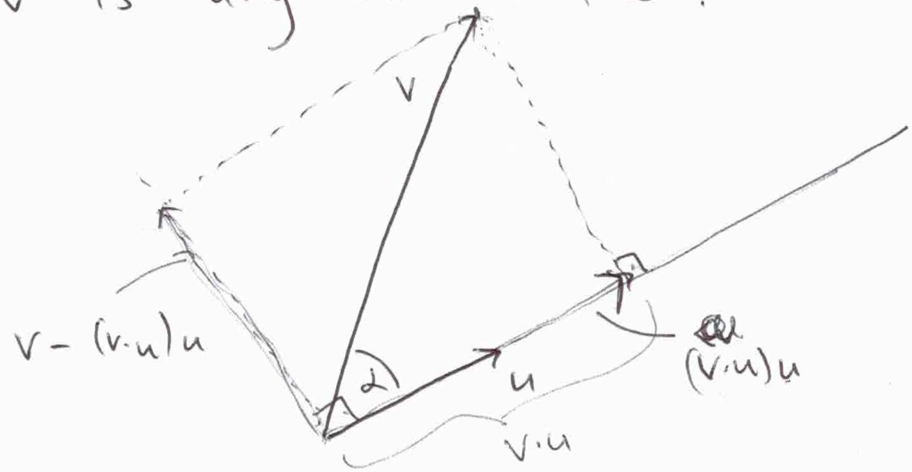
vectors are orthogonal/perpendicular

$$v \perp w \iff v \cdot w = 0.$$

Say u is a unit vector in $\mathbb{R}^2$.

$$\|u\| = 1 \iff \sqrt{u_1^2 + u_2^2} = 1 \iff u_1^2 + u_2^2 = 1$$
$$\iff u \cdot u = 1.$$

v is any vector. then:



$$\underline{\underline{v}} = (\underbrace{v \cdot u}_{\mathbb{R}})u + \big(v - (v\cdot u)u\big)$$

$\perp$ to each other.

$$\big((v\cdot u)u \cdot (v - (v\cdot u)u)\big)$$

$$= (v\cdot u)(u\cdot v) - (v\cdot u)(v\cdot u)\,1 = 0.$$

$$\boxed{\text{So } (v\cdot u)u \text{ is the orthogonal projection of } v \text{ onto the unit vector } u.}$$

$$\cos\alpha = \frac{v\cdot u}{\|v\|} \qquad \alpha = \cos^{-1}\frac{v\cdot u}{\|v\|} = \cos^{-1}\frac{v\cdot u}{\|v\|\,\|u\|}$$

$\alpha = \cos^{-1} \frac{v \cdot u}{\|v\| \, \|u\|}$ ← this formula, also true if u is not a unit vector.

$v \cdot u > 0 \Rightarrow \alpha$ angle is acute

$v \cdot u < 0 \Rightarrow \alpha$ is obtuse.

Definition: Let V be an $\mathbb{R}$-vector space.

An inner product on V is a map

$$V \times V \longrightarrow \mathbb{R}$$

$$(v,w) \longmapsto \langle v,w \rangle \quad \text{s.t.}$$

1. bilinear.
2. symmetric $\langle v,w\rangle = \langle w,v\rangle \quad \forall v,w \in V$.
3. positive definite: $\forall v \in V,\ v \neq 0$:

$$\underbrace{\langle v,v \rangle}_{\in \mathbb{R}} > 0.$$

↑ only makes sense in the field $\mathbb{R}$.

$\boxed{\begin{matrix} \langle v,0\rangle = 0 \\ \langle 0,w\rangle = 0 \end{matrix}}$

Suppose $\langle , \rangle$ is an inner product, define

$\|v\| = \sqrt{\langle v,v\rangle} \in \mathbb{R}_{\geq 0}$. Norm (length) of v.

$\|v\| = 0 \iff v = 0$.

A Euclidean VS is a real VS together w/ an inner product on V.

Example: $V = \mathbb{R}^n$ std VS.

$\langle , \rangle$ = dot product is the standard inner product.

Example. $V = \mathrm{Cont}([0,1], \mathbb{R})$.

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx .$$

Rmk. The Euclidean plane X (the plane of Euclidean geometry) is an affine space for a 2-dim'l real VS.

choose an origin $X \simeq V$

choose a basis in V. $V \approx \mathbb{R}^2$.

Let V be a Euclidean VS.

$v \in V$: $\quad \|v\| = \sqrt{\langle v, v\rangle}$. $\quad \boxed{\|v\|^2 = \langle v, v\rangle}$

angles: if $v, w \in V$, $v, w \neq 0$.

the angle α between v, w is defined to be $\quad \alpha = \cos^{-1} \dfrac{\langle v, w\rangle}{\|v\|\,\|w\|}$.

For this defn to make sense, need

$$\left| \frac{\langle v, w\rangle}{\|v\|\,\|w\|} \right| \leq 1.$$

$$|\langle v, w\rangle| \leq \|v\|\,\|w\|.$$ Cauchy-Schwarz inequality.

Proof. Case 1. $w = 0$. both sides $= 0$.

Case 2. $w \neq 0$. $\quad \lambda = \dfrac{\langle v, w\rangle}{\|w\|^2}$

$0 \leq \langle v - \lambda w, v - \lambda w\rangle$ (pos. def.)

(bilinear)

$= \langle v, v\rangle - \lambda \underline{\langle v, w\rangle}$

$\quad - \lambda \underline{\langle w, v\rangle} + \lambda^2 \langle w, w\rangle$

$= \langle v, v\rangle - 2\lambda\langle v, w\rangle + \lambda^2 \langle w, w\rangle$ (symmetry)

$\cancel{= \langle v, v\rangle} = \|v\|^2 - 2\dfrac{\langle v, w\rangle}{\|w\|^2}\langle v, w\rangle + \dfrac{\langle v, w\rangle^2}{\|w\|^{\cancel{4}2}}\cancel{\|w\|^2}$

$= \|v\|^2 - \dfrac{\langle v, w\rangle^2}{\|w\|^2}$

$$\frac{\langle v,w\rangle^2}{\|w\|^2} \leq \|v\|^2$$

$$\langle v,w\rangle^2 \leq \|v\|^2 \, \|w\|^2$$

$$|\langle v,w\rangle| \leq \|v\| \, \|w\| . \quad \square$$

Orthonormal Systems:

A family of mutually orthogonal unit vectors is called an ON system.

$(v_1, \dots, v_n)$: $\quad \langle v_i, v_j\rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Lemma. Every ON system is lin. indep.

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0.$$

apply the linear map $\langle v_i, \cdot \rangle$ to this eqn:

$$\lambda_1 \underbrace{\langle v_i, v_1\rangle}_{=0} + \dots + \lambda_i \underbrace{\langle v_i, v_i\rangle}_{=1} + \dots + \lambda_n \underbrace{\langle v_i, v_n\rangle}_{=0} = 0$$

$$\lambda_i = 0. \quad \square .$$

If an ON system spans V it is called an ON basis.

Suppose $(v_1, \dots, v_n)$ is an ON basis of V then $\forall v \in V$:

$$v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$$

$\underline{\text{Pf:}}$ since $(v_1, \dots, v_n)$ span V there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$:

$$v = \sum_{i=1}^{n} \lambda_i v_i \,.$$

pick another index j: apply $\underline{\langle v_j, \cdot \rangle}$:

$$\langle v_j, v \rangle = \sum_{i=1}^{n} \lambda_i \underbrace{\langle v_j, v_i \rangle}_{\substack{0 \text{ if } i \neq j \\ 1 \text{ if } i = j}}$$

$$= \lambda_j \qquad \square$$

orthogonal projections.

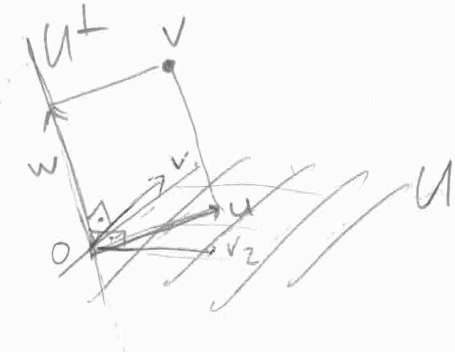
Let $(v_1, \dots, v_n)$ be an ON system in the Euclidean VS V.

$U = \mathrm{Span}(v_1, \dots, v_n)$ subspace in V.

$$U^\perp = \{ v \in V \mid \langle v, u \rangle = 0 \ \forall u \in U \}$$

$\dim V = \mathbb{R}$

$n = 2$



Every vector $v \in V$ can be written uniquely as $v = u + w$ with $u \in U, w \in U^\perp$.

$$\underbrace{u = \sum_{i=1}^n \langle v, v_i \rangle v_i}_{\in U}$$

the point is $u \in U$, $v - u \in U^\perp$.

(check as in the very beginning)

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto \sum_{i=1}^n \langle v, v_i \rangle v_i \end{aligned}$$

is the orthogonal projection of V onto U.
linear map.