Affine Spaces

let V be a vector space over the field | field

field $\mathbb{F}$.

Defn: An affine space for V is a set X together with an operation

$$+ : X \times V \longrightarrow X$$
$$(x,v) \longmapsto x+v$$

Such that: (1) $(x+v)+w = x+(v+w)$ $\forall x \in X$ $\forall v,w \in V$

add. in V

(2) $x+0 = x$ $\forall x \in X$

(3) $\forall x,y \in X$ $\exists$ unique $v \in V$; s.t. $y = x+v$.

Examples: (1) durations & dates

$\mathbb{F} = \mathbb{R}$.

D: durations (5 minutes, 10 years, -2 seconds)

you can add durations:

$$D \times D \longrightarrow D$$
$$(x,y) \longmapsto x+y$$

multiply durations by real numbers

$\mathbb{R} \times D \longrightarrow D$.

D: durations form a real vector space.

$\dim D = 1$. (for example, 1 year is a basis of D)

X: dates (e.g. Jan 1 2009, 10 am March 18, ...)

you can add durations to dates to get dates:

$x =$ Jan 1, 2009

$d =$ 5 days

$x + d =$ Jan 6, 2009.

have an operation

$$X \times D \longrightarrow X$$

$$(x, d) \longmapsto x + d.$$

X: dates
D: durations

it makes no sense to ~~dates~~ add dates to each other.

the dates form a affine space for ~~the~~ the durations.

Axioms: $(x+d)+e = x+(d+e)$ ✓

$(x+0) = x$ ✓

$\uparrow$ 0 duration

for any two dates x, y there is a unique duration d s.t.

$x+d=y$. ✓

Often physical applications involve affine spaces.

Example ② displacements & locations.

~~location~~ Assume flat earth.

locations:
- Vancouver
- your seat in LSK 460

displacements:
- 4 miles north
- 3 miles east
- 3 blocks west & 2 blocks south.

the displacements form a 2-dim'l $\mathbb{R}$-VS.

basis: e.g. 1 mile north
1 mile west.

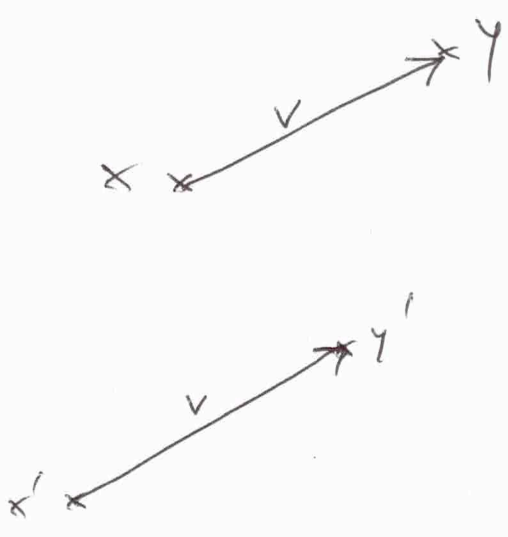
the locations are an affine space for the displacements.

For any two locations there is a unique displacement going from the 1st location to second.

<u>locations</u>:

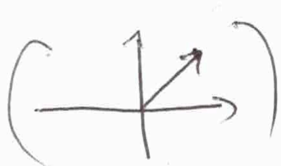
x, y: location

$\nearrow$: displacement.



$x + v = y$
$x' + v = y'$.

often vector spaces occur as vector spaces of affine spaces.



If X is an affine space for the vector space V, we can define:

$$X \times X \longrightarrow V$$

$$(x,y) \longmapsto \text{the unique } v \text{ s.t. } x+v=y.$$

we write $v = y-x$

$$X \times X \longrightarrow V$$

$$(x,y) \longmapsto y-x.$$

Subtract elements in the affine space to get vectors.

Example (3) Solution sets to inhomogeneous systems of equations:

Consider $A\vec{x} = \vec{b}$.

$A \in M(m \times n, \mathbb{F})$ coeff. matrix.

n: # cols = # indeterminats.

Solution set is a subset of $\mathbb{F}^n$.

m = # of equations.

$\vec{b} \in \mathbb{F}^m$ augmentation column.

augmented matrix: $(A \mid \vec{b})$.

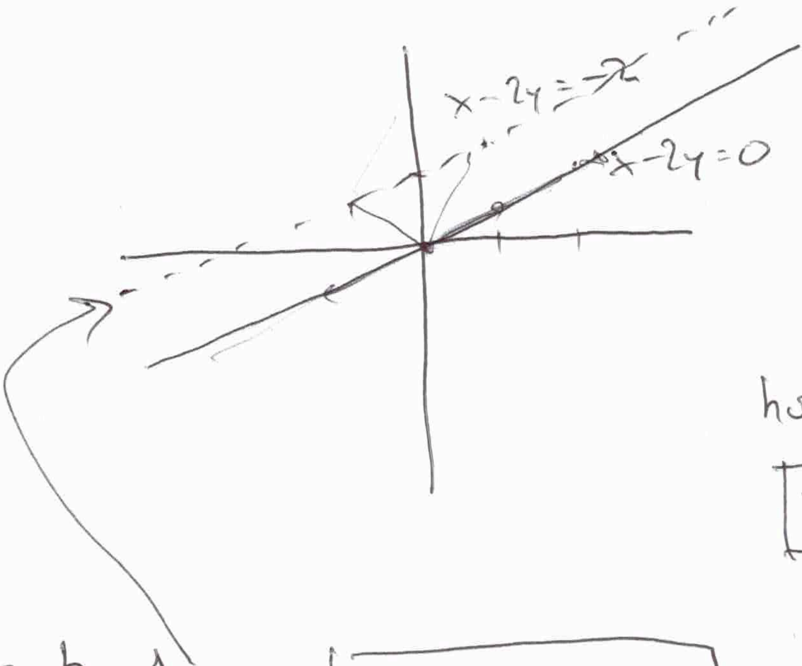
Let $A\vec{x} = \vec{0}$ be the assoc. homog. system, and $V \subset \mathbb{F}^n$ the solution space, is a subspace.

$V = Nul(A) = \ker(L_A)$.

Consider $A\vec{x} = \vec{b}$. then

S = Soln set is an affine space for V.

$\mathbb{F} = \mathbb{R}$



$V = \operatorname{span}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \ker(1 \ \ -2)$.

$A = (1 \ \ -2)$

homog. eqtn is

$\boxed{x - 2y = 0.}$

eqtn of

$\boxed{x - 2y = -2}$

$x = 0$

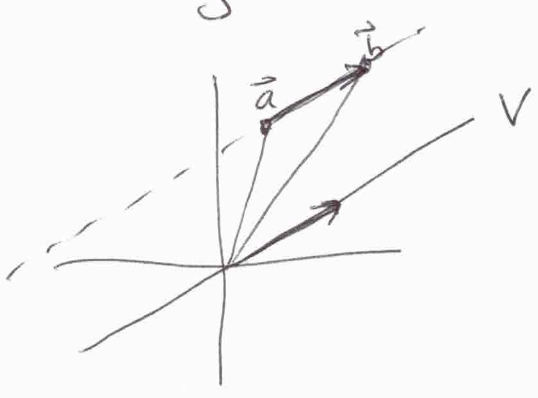
$y = 1$

S = Soln set to $x - 2y = -2$

is an affine space for

V = soln set of $x - 2y = 0$.

for any ~~to~~ two solutions of $x-2y=-2$



the difference $\vec{b} - \vec{a}$

is a soln of the

homog. system.

~~the~~

$$V = \{\vec{x} \in \mathbb{F}^n \mid A\vec{x} = \vec{0}\}$$

$$X = \{\vec{x} \in \mathbb{F}^n \mid A\vec{x} = \vec{b}\}$$

then X is an affine space for V.

$$X \times V \longrightarrow X$$
$$(\vec{x}, \vec{v}) \longmapsto \vec{x} + \vec{v}$$

add in $\mathbb{F}^n$.

well-defined b/c $A\vec{x} = \vec{b}$

$A\vec{v} = \vec{0}$

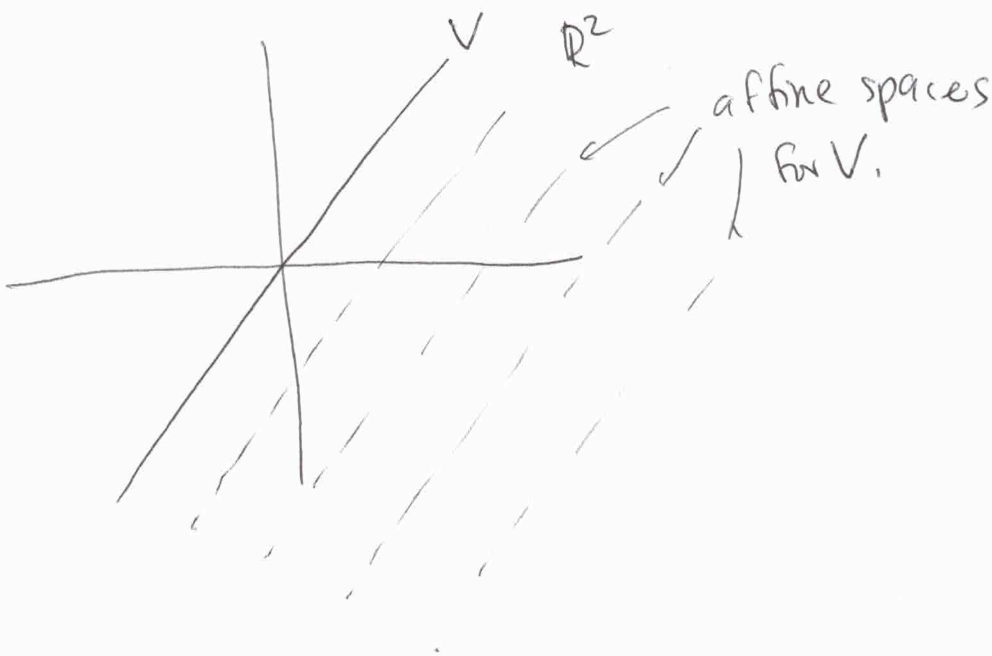
$$A(\vec{x}+\vec{v}) =$$
$$A\vec{x} + A\vec{v} = \vec{b} + \vec{0}$$
$$= \vec{b}.$$

If $\vec{x}, \vec{y}$ are both in X:

$$A\vec{x} = \vec{b}$$
$$A\vec{y} = \vec{b}$$

then $A(\vec{y} - \vec{x}) = \vec{0}$

$$X \times X \longrightarrow V.$$
$$(\vec{x}, \vec{y}) \longmapsto \vec{y} - \vec{x}$$



An affine space is like a vector space without an origin.

If you pick a fixed base point in an affine space: $x_0 \in X$

for example choose a date:

Jesus' birthday.

then

$$V \xrightarrow{\Phi} X$$

$$v \longmapsto x_0 + v$$

~~the~~ is a bijection!

and the addition in V corresponds to the affine operation.

$x_0 \in X$ fixed.

$$V \xrightarrow{\Phi} X \qquad v \longmapsto x_0 + v.$$

is a bijection by the axioms of affine space.
($\forall x \in X \ \exists$ unique $v \in V$ s.t. $\Phi(v) = x$.
i.e. $x_0 + v = x$)

$\cancel{\Phi(v+w)}$

$$\Phi(v+w) = x_0 + (v+w)$$
$$= (x_0 + v) + w$$
$$= \Phi(v) + w$$

addition in V

affine operation of V on X.

the choice of $x_0 \in X$ allows us to identify V and X.

e.g. choose a fixed date: Jan 1, 0.

then $\{\text{durations}\} \longrightarrow \{\text{dates}\}$

$$5 \text{ years} \longmapsto \text{Jan } 1, 05$$

<u>locations:</u> (affine space for displacements).

affine spaces: vector spaces w/o origin.