Practice Exam

Problem 1. (0 points)
Suppose $T : \mathbb{R}^4 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^5$ are linear maps.

(a) What is the largest possible value for the rank of the composition $S \circ T : \mathbb{R}^4 \to \mathbb{R}^5$?

(b) Give explicit examples for S and T where this maximal value is attained.

(c) Explain why the rank cannot be any larger than your answer to (a).

Problem 2. (0 points)
Find general solutions to the systems of differential equations $\vec{x}\,'(t) = A\,\vec{x}(t)$, for the following transition matrices A. Always give your answer in terms of real numbers.

(a)

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(d)

$$A = \begin{pmatrix} 2 & -3 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

Problem 3. (0 points)
Four of King Arthur's knights sit around the round table, and are served unequal amounts of cereal. Every time the gong sounds, each knight takes half of the cereal from both of his neighbours. Determine the distribution of cereal in the knight's bowls after 10 and 15 sounds of the gong.

Problem 4. (0 points)
Sketch phase portraits of the following discrete dynamical systems:

(a)

$$\vec{x}_{n+1} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \vec{x}_n$$

(b)

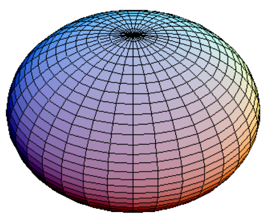
$$\vec{x}_{n+1} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \vec{x}_n$$

Problem 5. (0 points)
Find the principal axes of the quadric

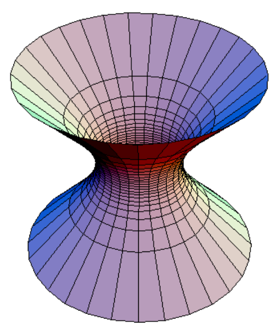
$$4x^2 + 4y^2 + 4z^2 + 2xy + 2xz + 2yz = 1\,.$$

Which one of the following three types is it?

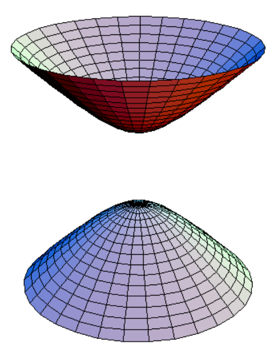
(a) Ellipsoid



(b) Hyperboloid of one sheet



(c) Hyperboloid of two sheets



Problem 6. (0 points)
Find a change of coordinates $\vec{x} = P\vec{x}\,'$ which removes the cross-terms from the quadratic form

$$Q(x, y, z) = 3x^2 + 4y^2 + 5z^2 + 2xy + 2x + 2yz\,.$$

Problem 7. (0 points)
Find bases for nullspace, row space and column space of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & t \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & t & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & 3 & 4 & 0 & 1 \end{pmatrix}$$

Be careful to explain how the result depends on the value of t.

Problem 8. (0 points)
Find an orthonormal basis of the nullspace of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 2 & 3 & 0 & 1 & 0 \end{pmatrix}$$

Problem 9. (0 points)
Find a basis of the nullspace of

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} .$$

Then complete this to a basis of $\mathbb{R}^4$ with vectors from the list

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

Problem 10. (0 points)
Are the following matrices similar?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Problem 11. (0 points)
Put the following matrices into classes of similar ones.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}$$
$$\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

Problem 12. (0 points)
Find Jordan canonical forms of the matrices

$$\begin{pmatrix}1&2&3\\0&2&3\\0&0&3\end{pmatrix} \quad \begin{pmatrix}2&0&3\\0&2&1\\0&0&3\end{pmatrix} \quad \begin{pmatrix}3&2&1\\0&3&0\\0&0&3\end{pmatrix}$$

Problem 13. (0 points)
The subspace $V \subset \mathbb{R}^4$ is spanned by $\left(\begin{smallmatrix}1\\0\\1\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1\\1\\0\\1\end{smallmatrix}\right)$. The subspace $W \subset \mathbb{R}^4$ is spanned by $\left(\begin{smallmatrix}2\\0\\2\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\\-1\\2\end{smallmatrix}\right)$. Find the dimension of $V \cap W$, and a basis for $V \cap W$.

Problem 14. (0 points)
(a) Find the determinant of the matrix

$$\begin{pmatrix}0&1&1&0&0\\t&0&0&1&2\\u&0&0&t&1\\0&1&1&0&1\\2&2&0&2&0\end{pmatrix}$$

(b) What conditions do t and u have to satisfy in order for this matrix to be invertible?

Problem 15. (0 points)
Let A be a real symmetric matrix with eigenvalues 2 and 6. Suppose that the eigenspace of 6 is spanned by $\left(\begin{smallmatrix}1\\1\\-1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right)$. Find A.

Problem 16. (0 points)
Find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Problem 17. (0 points)
Suppose A is a real 2×2 matrix. Prove that A and its transpose are similar to each other.

Problem 18. (0 points)
A co-op produces four types of cookies: A, B, C, D. Ingredients needed to make one box of cookies of each type are given in the table below:

	A	B	C	D
flour(cups)	3	6	3	9
butter(lbs)	1	2	1	3
sugar(cups)	2	1	3	8
eggs	1	4	2	7
chocolate(lbs)	0	3	1	2

During one hour of operation, the following amounts of ingredients were used: 33 cups of flour, 11 lbs of butter, 23 cups of sugar, 21 eggs and 7 lbs of chocolate. Find how many boxes of each type were produced.

Problem 19. (0 points)
Find the determinant of the $n \times n$ matrix with non-zero entries along the three diagonals only:

$$\begin{pmatrix} 6 & 1 & & & & \\ 5 & 6 & 1 & & & \\ & 5 & 6 & 1 & & \\ & & & \cdots & & \\ & & & 5 & 6 & 1 \\ & & & & 5 & 6 \end{pmatrix}$$

Problem 20. (0 points)
Suppose the subspaces V of dimension 2 and $V^{\perp}$ of dimension 3 in $\mathbb{R}^5$ (standard inner product) are eigenspaces of the linear operator $T : \mathbb{R}^5 \to \mathbb{R}^5$. Prove that the standard basis of T is symmetric.