Problem 7

Put the matrix into reduced row echelon form using Gaussian elimination:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & t \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & t & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & 3 & 4 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ -tI \\ -I \\ -I \end{matrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & t \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1-t & 0 & 1 & 1-t^2 \\ 0 & 0 & 1 & 2 & 0 & 2-t \\ 0 & 0 & 2 & 4 & 0 & 1-t \end{pmatrix} \begin{matrix} \\ \\ +(t-1)II \\ -II \\ -2II \end{matrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & t \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2t-2 & 1 & 1-t^2 \\ 0 & 0 & 0 & 0 & 0 & 2-t \\ 0 & 0 & 0 & 0 & 0 & 1-t \end{pmatrix}$$

Case $\boxed{A}$ $t=1$:

$$\underbrace{\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{ref}} \rightarrow \underbrace{\begin{pmatrix} 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{rref}}$$

nullspace: free variables x_1, x_4 set equal 1,0 and 0,1 to get basis $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

column space: pivot columns 2, 3, 5 6 of original matrix basis: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$

row space: pivot rows of final matrix $(0\ 1\ 0\ -2\ 0\ 0), (0\ 0\ 1\ 2\ 0\ 0), (0\ 0\ 0\ 0\ 1\ 0), (0\ 0\ 0\ 0\ 0\ 1)$.

Case $\boxed{B}$ $t \neq 1$: multiply rows III and V by $\frac{1}{t-1}$ to get

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & t \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2(t-1)} & -\frac{1}{2}(t+1) \\ 0 & 0 & 0 & 0 & 0 & 2-t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\underbrace{\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2(t-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{ref}} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{1-t} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2(t-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & \frac{1}{t-1} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{1-t} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2(t-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{rref}$$

null space: free vars x_1, x_5 set equal 1,0 and 0,1 to get $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{1-t} \\ \frac{1}{t-1} \\ \frac{1}{2(1-t)} \\ 1 \\ 0 \end{pmatrix}$

or, another basis: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -2 \\ 1 \\ 2-2t \\ 0 \end{pmatrix}$.

column space: pivot cols 2, 3, 4, 6: $\begin{pmatrix} 1 \\ 0 \\ t \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} t \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$

(Rmk: removing column #4 from rref, we see that 2, 3, 5, 6 are also linearly indep and form a basis of the column space.)

row space: basis
$(0 \ 1 \ 0 \ 0 \ \frac{1}{t-1} \ 0)$
$(0 \ 0 \ 1 \ 0 \ \frac{1}{1-t} \ 0)$
$(0 \ 0 \ 0 \ 1 \ \frac{1}{2(t-1)} \ 0)$
$(0 \ 0 \ 0 \ 0 \ 0 \ 1)$

or, getting rid of denominators:
$(0 \ \ t-1 \ \ 0 \ \ 0 \ \ 1 \ \ 0)$
$(0 \ \ 0 \ \ t-1 \ \ 0 \ \ -1 \ \ 0)$
$(0 \ \ 0 \ \ 0 \ \ 2t-2 \ \ 1 \ \ 0)$
$(0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 1)$

Problem 8. Find basis vectors one at a time:

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 2 & 3 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & -1 & -6 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -9 & 2 & -3 \\ 0 & 1 & 6 & -1 & 2 \end{pmatrix}$$

$x_3 = 1$ $x_4 = 0$ $x_5 = 0$ gives $\vec{v}_1 = \begin{pmatrix} 9 \\ -6 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

the other basis vectors need to be $\perp \vec{v}_1$, so put $\vec{v}_1$ as extra equation:

$$\begin{pmatrix} 1 & 0 & -9 & 2 & -3 \\ 0 & 1 & 6 & -1 & 2 \\ 9 & -6 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -9 & 2 & -3 \\ 0 & 1 & 6 & -1 & 2 \\ 0 & -6 & 82 & -18 & 27 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -9 & 2 & -3 \\ 0 & 1 & 6 & -1 & 2 \\ 0 & 0 & 118 & -24 & 39 \end{pmatrix}$$

these numbers are getting unpleasant. Let's try a different $\vec{v}_1$.

$x_3 = 0$ $x_4 = 1$ $x_5 = 0$ gives $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ put this as extra equation:

$$\begin{pmatrix} 1 & 0 & -9 & 2 & -3 \\ 0 & 1 & 6 & -1 & 2 \\ -2 & 1 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -9 & 2 & -3 \\ 0 & 1 & 6 & -1 & 2 \\ 0 & 1 & -18 & 5 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -9 & 2 & -3 \\ 0 & 1 & 6 & -1 & 2 \\ 0 & 0 & -24 & 6 & -8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -9 & 2 & -3 \\ 0 & 1 & 6 & -1 & 2 \\ 0 & 0 & 1 & -1/4 & 1/3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & 1/3 \end{pmatrix}$$

put $x_4 = 0$ $x_5 = 3$ (to keep numbers small) $\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 3 \end{pmatrix}$

put as extra equation:

$$\begin{pmatrix} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & 1/3 \\ 0 & 0 & -1 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & 1/3 \\ 0 & 0 & 0 & -1/4 & 10/3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & 1/3 \\ 0 & 0 & 0 & 1 & -40/3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 & -10/3 \\ 0 & 1 & 0 & 0 & 20/3 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -40/3 \end{pmatrix}$$

put $x_5 = 3$, get $\begin{pmatrix} 10 \\ -20 \\ 9 \\ 40 \\ 3 \end{pmatrix}$

So an orthogonal basis of the null space is

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} 10 \\ -20 \\ 9 \\ 40 \\ 3 \end{pmatrix}$$

normalize to get an ON basis:

$$\sqrt{4+1+1} = \sqrt{6}$$

$$\sqrt{1+9} = \sqrt{10}$$

$$\sqrt{10^2+20^2+9^2+40^2+3^2} = \sqrt{100+400+81+1600+9} = \sqrt{2190}$$

ON basis:

$$\frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{10}}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \quad \frac{1}{\sqrt{2190}}\begin{pmatrix} 10 \\ -20 \\ 9 \\ 40 \\ 3 \end{pmatrix}.$$

Problem 9.

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$\boxed{\text{basis of nullspace} \quad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

now want to pick a linearly indep set from $\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$,

but making sure that $\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ stay on the list.

If we put them all as columns with $\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ <u>first</u> this will be achieved:

$$\begin{pmatrix} -3 & -1 & 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 2 & 0 & 1 \\ -3 & -1 & 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 & 3 & 3 \end{pmatrix} \begin{matrix} \\ +3\mathrm{I} \\ -\mathrm{I} \\ \\ \end{matrix} \to \begin{pmatrix} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & -1 & 3 & 7 & 2 & 5 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 0 & 3 & 3 \end{pmatrix} \begin{matrix} \\ \\ \\ +\mathrm{II} \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & -3 & -7 & -2 & -5 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 6 & 7 & 5 & 8 \end{pmatrix} \begin{matrix} \\ \\ \\ -6\,\mathrm{III} \end{matrix} \quad \begin{pmatrix} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & -3 & -7 & -2 & -5 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 14 \end{pmatrix}$$

the first 4 are linearly indep.

$$\boxed{\text{basis completed to basis of } \mathbb{R}^4. \quad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}}$$

Problem 10

The second matrix $\begin{pmatrix} 1 & 1 & & \\ & 2 & 1 & \\ & & 3 & 1 \\ & & & 4 \end{pmatrix}$ has 4 distinct eigenvalues, $1, 2, 3, 4$.

So it is diagonalizable, there exists invertible matrix P such that

$$\begin{pmatrix} 1 & 1 & & \\ & 2 & 1 & \\ & & 3 & 1 \\ & & & 4 \end{pmatrix} = P \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} P^{-1}.$$

So the 2 matrices given are similar, <u>YES</u>.

Problem 11.

Find Jordan canonical forms by comparing algebraic & geometric multiplicities of eigenvalues. All eigenvalues of all matrices given are $\lambda = 1$, each time with algebraic multiplicity 3. The geometric multiplicity will give the number of Jordan cells in the Jordan canonical form.

For 3x3 matrices there are 3 possibilities:

geom. mult $= 3 \longrightarrow 3$ Jordan cells: $\begin{pmatrix} \boxed{1} & 0 & 0 \\ & \boxed{1} & 0 \\ & & \boxed{1} \end{pmatrix}$ write $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

geom mult $= 2 \longrightarrow 2$ Jordan cells: $\begin{pmatrix} \boxed{1} & 0 & 0 \\ & \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} \end{pmatrix}$ write $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

geom mult $= 1 \longrightarrow 1$ Jordan cell: $\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{matrix}} \end{pmatrix}$. write $A_3 = \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

geom. mult $=$ dim(eigenspace) $=$ dim Nul$(A - \lambda I)$ $=$ dim Nul$(A - I)$ for $\lambda = 1$.

$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} : \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$ geom mult 3 $\rightarrow$ similar to A_1

$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} : \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ geom mult 2 $\rightarrow$ similar to A_2

$\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} : \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ - - - - - - - - - - - A_2

$\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix} : \begin{pmatrix} 0 & 1 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ - - - - - - - - - - - A_2

Problem 11 (cont'd)

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ geom mult 2 → similar to A_2

$\begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix} : \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ ---------- A_2

$\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} : \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$ geom mult = 1 → similar to A_3

$\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix} : \begin{pmatrix} 0 & 1 & 1 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .... A_3 .

Problem 12. Find Jordan forms:

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$ 3 distinct eigenvalues, so 3 Jordan cells of size 1: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is Jordan form.

$\begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ eigenvalues 2, 3. need the geometric multiplicity of 2:

$A-2I = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ geom. mult = 2, so 2 Jordan cells with eigenvalue 2.

Jordan form: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ one eigenvalue, 3. $A-3I = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1/2 \end{pmatrix}$

geom. mult = 2, so 2 Jordan cells: $\boxed{\begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}}$ and $\boxed{3}$:

Jordan form is $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ [or $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$] .

Problem 13

parametric form of V is $\vec{x} = a\begin{pmatrix}1\\0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$

" " " W is $\vec{x} = c\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + d\begin{pmatrix}0\\1\\-1\\2\end{pmatrix}$.

to find intersection, set them equal: $a\begin{pmatrix}1\\0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\1\\0\\1\end{pmatrix} - c\begin{pmatrix}2\\0\\2\\0\end{pmatrix} - d\begin{pmatrix}0\\1\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

or $\begin{pmatrix}1 & 1 & -2 & 0\\0 & 1 & 0 & -1\\1 & 0 & -2 & 1\\1 & 1 & 0 & -2\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.

Solve using Gauss:

$$\begin{pmatrix}1 & 1 & -2 & 0\\0 & 1 & 0 & -1\\1 & 0 & -2 & 1\\1 & 1 & 0 & -2\end{pmatrix} \to \begin{pmatrix}1 & 1 & -2 & 0\\0 & 1 & 0 & -1\\0 & -1 & 0 & 1\\0 & 0 & 2 & -2\end{pmatrix} \to \begin{pmatrix}1 & 1 & -2 & 0\\0 & 1 & 0 & -1\\0 & 0 & 0 & 0\\0 & 0 & 1 & -1\end{pmatrix} \to \begin{pmatrix}1 & 1 & 0 & -2\\0 & 1 & 0 & -1\\0 & 0 & 1 & -1\end{pmatrix}$$

$$\to \begin{pmatrix}1 & 0 & 0 & -1\\0 & 1 & 0 & -1\\0 & 0 & 1 & -1\end{pmatrix}$$ parametric solution: $\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}t\\t\\t\\t\end{pmatrix}$

So the most general vector in $V \cap W$ is $\vec{x} = t\begin{pmatrix}1\\0\\1\\1\end{pmatrix} + t\begin{pmatrix}1\\1\\0\\1\end{pmatrix} = t\begin{pmatrix}2\\1\\1\\2\end{pmatrix}$ using a, b

or $\vec{x} = t\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\-1\\2\end{pmatrix} = t\begin{pmatrix}2\\1\\1\\2\end{pmatrix}$ using c, d.

we see that the vector $\begin{pmatrix}2\\1\\1\\2\end{pmatrix}$ is a basis for $V \cap W$ so $\dim V \cap W = 1$.

Problem 14

$$\begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ t & 0 & 0 & 1 & 2 \\ u & 0 & 0 & t & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 & 0 \end{vmatrix} \underset{\text{column operation}}{=} \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ t & 0 & 0 & 1 & 2 \\ u & 0 & 0 & t & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 & 0 \end{vmatrix} \underset{\text{Laplace expansion}}{=}$$

$-\mathrm{III}$

$$= -2 \begin{vmatrix} 0 & 1 & 0 & 0 \\ t & 0 & 1 & 2 \\ u & 0 & t & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} \underset{\text{Laplace expansion}}{=} +2 \begin{vmatrix} t & 1 & 2 \\ u & t & 1 \\ 0 & 0 & 1 \end{vmatrix} \underset{\text{Laplace expansion}}{=} 2 \begin{vmatrix} t & 1 \\ u & t \end{vmatrix} = 2(t^2-u).$$

the matrix is invertible if $u \neq t^2$.

Problem 15

A is orthogonally diagonalizable. So $A = PDP^t$ where P is an orthogonal matrix whose columns consist of eigenvectors for A.

We know E_6 (eigenspace of 6) is spanned by $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Luckily, these 2 vectors are orthogonal. We know that E_2 (eigenspace of 2) is $\perp E_6$. So if $\vec{v}$ is an eigenvector w/ eigenvalue 2, then $\vec{v} \perp \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\vec{v} \perp \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. This allows us to find $\vec{v}$:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \end{pmatrix}$$

So we can take $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

So we have an orthogonal basis of eigenvectors: $\underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\lambda=6} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}_{\lambda=2}$.

normalize, to get ON basis:

$$\frac{\sqrt{3}}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \frac{\sqrt{6}}{6}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Problem 15 (cont'd)

So we have $P = \frac{1}{6}\begin{pmatrix} 2\sqrt{3} & 3\sqrt{2} & \sqrt{6} \\ 2\sqrt{3} & -3\sqrt{2} & \sqrt{6} \\ -2\sqrt{3} & 0 & 2\sqrt{6} \end{pmatrix}$

and $P^{-1} = P^t = \frac{1}{6}\begin{pmatrix} 2\sqrt{3} & 2\sqrt{3} & -2\sqrt{3} \\ 3\sqrt{2} & -3\sqrt{2} & 0 \\ \sqrt{6} & \sqrt{6} & 2\sqrt{6} \end{pmatrix}$

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

so $A = PDP^t = \frac{1}{36}\begin{pmatrix} 2\sqrt{3} & 3\sqrt{2} & \sqrt{6} \\ 2\sqrt{3} & -3\sqrt{2} & \sqrt{6} \\ -2\sqrt{3} & 0 & 2\sqrt{6} \end{pmatrix}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2\sqrt{3} & 2\sqrt{3} & -2\sqrt{3} \\ 3\sqrt{2} & -3\sqrt{2} & 0 \\ \sqrt{6} & \sqrt{6} & 2\sqrt{6} \end{pmatrix}$

$$= \frac{1}{36}\begin{pmatrix} 12\sqrt{3} & 18\sqrt{2} & 2\sqrt{6} \\ 12\sqrt{3} & -18\sqrt{2} & 2\sqrt{6} \\ -12\sqrt{3} & 0 & 2\sqrt{6} \end{pmatrix}\begin{pmatrix} 2\sqrt{3} & 2\sqrt{3} & -2\sqrt{3} \\ 3\sqrt{2} & -3\sqrt{2} & 0 \\ \sqrt{6} & \sqrt{6} & 2\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} 16/3 & -2/3 & -4/3 \\ -2/3 & 16/3 & -4/3 \\ -4/3 & -4/3 & 8/3 \end{pmatrix} = \boxed{\frac{2}{3}\begin{pmatrix} 8 & -1 & -2 \\ -1 & 8 & -2 \\ -2 & -2 & 4 \end{pmatrix}}$$

Problem 16

$$\left(\begin{array}{cccc|cccc} 1&1&0&0&1&0&0&0\\ 1&1&1&0&0&1&0&0\\ 0&1&1&1&0&0&1&0\\ 0&0&1&1&0&0&0&1 \end{array}\right) \xrightarrow{-\mathrm{I}} \left(\begin{array}{cccc|cccc} 1&1&0&0&1&0&0&0\\ 0&0&1&0&-1&1&0&0\\ 0&1&1&1&0&0&1&0\\ 0&0&1&1&0&0&0&1 \end{array}\right) \leftrightarrow \left(\begin{array}{cccc|cccc} 1&1&0&0&1&0&0&0\\ 0&1&1&1&0&0&1&0\\ 0&0&1&0&-1&1&0&0\\ 0&0&1&1&0&0&0&1 \end{array}\right) -\mathrm{III}$$

$$\to \left(\begin{array}{cccc|cccc} 1&1&0&0&1&0&0&0\\ 0&1&1&1&0&0&1&0\\ 0&0&1&0&-1&1&0&0\\ 0&0&0&1&1&-1&0&1 \end{array}\right) \xrightarrow{-\mathrm{III}\ -\mathrm{IV}} \left(\begin{array}{cccc|cccc} 1&1&0&0&1&0&0&0\\ 0&1&0&0&0&0&1&-1\\ 0&0&1&0&-1&1&0&0\\ 0&0&0&1&1&-1&0&1 \end{array}\right) \xrightarrow{-\mathrm{II}} \left(\begin{array}{cccc|cccc} 1&0&0&0&1&0&-1&1\\ 0&1&0&0&0&0&1&-1\\ 0&0&1&0&-1&1&0&0\\ 0&0&0&1&1&-1&0&1 \end{array}\right)$$

$$\begin{pmatrix} 1&1&0&0\\ 1&1&1&0\\ 0&1&1&1\\ 0&0&1&1 \end{pmatrix}^{-1} = \begin{pmatrix} 1&0&-1&1\\ 0&0&1&-1\\ -1&1&0&0\\ 1&-1&0&1 \end{pmatrix}$$

Problem 18

The table describes a matrix transformation

$$\mathbb{R}^4 = \text{cookie box vectors} \longrightarrow \text{ingredient vectors} = \mathbb{R}^5$$

$$\begin{pmatrix} A\\B\\C\\D \end{pmatrix} \longmapsto \begin{pmatrix} f\\b\\s\\e\\c \end{pmatrix}$$

$$\begin{pmatrix} f\\b\\s\\e\\c \end{pmatrix} = \begin{pmatrix} 3&6&3&9\\ 1&2&1&3\\ 2&1&3&8\\ 1&4&2&7\\ 0&3&1&2 \end{pmatrix} \begin{pmatrix} A\\B\\C\\D \end{pmatrix}$$

an ingredients vector $\begin{pmatrix} 33\\11\\23\\21\\7 \end{pmatrix}$ is given. We want to find all possible cookie box vectors $\begin{pmatrix} A\\B\\C\\D \end{pmatrix}$.

this amounts to solving $\begin{pmatrix} 3&6&3&9\\ 1&2&1&3\\ 2&1&3&8\\ 1&4&2&7\\ 0&3&1&2 \end{pmatrix} \begin{pmatrix} A\\B\\C\\D \end{pmatrix} = \begin{pmatrix} 33\\11\\23\\21\\7 \end{pmatrix}$. Use Gauss

$$\left(\begin{array}{cccc|c} 3&6&3&9&33\\ 1&2&1&3&11\\ 2&1&3&8&23\\ 1&4&2&7&21\\ 0&3&1&2&7 \end{array}\right) \to \left(\begin{array}{cccc|c} 1&0&0&0&3\\ 0&1&0&0&1\\ 0&0&1&0&0\\ 0&0&0&1&2\\ 0&0&0&0&0 \end{array}\right)$$

there is a unique solution. $A=3$, $B=1$, $C=0$, $D=2$ are the numbers of boxes of the 4 types of cookies produced.

Problem 17.

Recall that a real 2×2 matrix is similar to exactly one of 4 types:

① $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ if A has two distinct real eigenvalues $\lambda_1 \neq \lambda_2$.

② $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ if A has two distinct cplx eigenvalues $\lambda = a \pm ib$

③ $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ if A is diagonal

④ $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ if A has only one real eigenvalue and A is <u>not</u> diagonal.

Now A and A^t have the same characteristic polynomial, so the same eigenvalues.

Also, A is diagonal if and only if A^t is diagonal.

So the above classification is in terms of information which is the same for A and A^t.

So no matter what A is, A and A^t have to fall into the same case ①, ②, ③ or ④.

So A and A^t are similar.

Problem 20.

In every subspace of $\mathbb{R}^5$ we can choose an ON basis. So let $\vec{u}_1, \vec{u}_2$ be an ON basis of V and $\vec{u}_3, \vec{u}_4, \vec{u}_5$ an ON basis of $V^\perp$. Then $\vec{u}_1$ and $\vec{u}_2$ are orthogonal to $\vec{u}_3, \vec{u}_4, \vec{u}_5$ so $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5$ is an ON basis of $\mathbb{R}^5$. Let P be the matrix with $\vec{u}_1, \dots, \vec{u}_5$ as columns. Then P is orthogonal, so $P^{-1} = P^t$.

Let λ be the eigenvalue of T on V and μ the eigenvalue of T on $V^\perp$.

Then $[T]_{\mathcal{B}} = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \mu & & \\ & & & \mu & \\ & & & & \mu \end{pmatrix}$ if $\mathcal{B}$ is the basis $\vec{u}_1, \dots, \vec{u}_5$. Write $D = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \mu & & \\ & & & \mu & \\ & & & & \mu \end{pmatrix}$.

Then $A = PDP^{-1} = PDP^t$ and $A^t = (PDP^t)^t = (P^t)^t D^t P^t = PDP^t = A$ $\square$

$\boxed{\text{if } A = [T]_{\mathcal{E}}}$

Problem 19

Let A_n be the given matrix of size $n \times n$ and $a_n = \det A_n$.

$A_0 = \emptyset \qquad A_1 = (6) \qquad A_2 = \begin{pmatrix} 6 & 1 \\ 5 & 6 \end{pmatrix}$ etc.

$a_0 = 1 \qquad a_1 = 6 \qquad a_2 = 36 - 5 = 31$ etc.

For A_n:

$$a_n = \det \begin{pmatrix} 6 & 1 & \\ 5 & \begin{array}{|cc} 6 & 1 \\ 5 & A_{n-1} \end{array} \end{pmatrix} = 6a_{n-1} - \det \begin{pmatrix} 5 & 1 \\ 0 & \begin{array}{|c} A_{n-2} \end{array} \end{pmatrix} \quad \text{by expanding the } 1^{st} \text{ row}$$

$$= 6a_{n-1} - 5a_{n-2}$$

[A careful analysis reveals that this formula holds as soon as $n \geq 2$, with the above conventions for a_0, a_1.]

To solve the recursion $a_{n+1} = 6a_n - 5a_{n-1}$ substitute $b_n = a_{n-1}$, so that only the "times" n and $n+1$ are involved:

$$a_{n+1} = 6a_n - 5b_n \qquad \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 6 & -5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$
$$b_{n+1} = a_n$$

transition matrix $A = \begin{pmatrix} 6 & -5 \\ 1 & 0 \end{pmatrix}$. $\det(\lambda I - A) = \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5)$

$\underline{\underline{\lambda = 1}}$ $\quad A - I = \begin{pmatrix} 5 & -5 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\underline{\underline{\lambda = 5}}$ $\quad A - 5I = \begin{pmatrix} 1 & -5 \\ 1 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -5 \\ 0 & 0 \end{pmatrix}$ eigenvector $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$

gen'l solution $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = c_1 (1)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 5^n \begin{pmatrix} 5 \\ 1 \end{pmatrix}$. initial condition $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 6 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 5c_2 \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 25 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & 25 & 6 \\ 1 & 5 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & -1/4 \\ 0 & 1 & +1/4 \end{array}\right)$

$$a_n = -\frac{1}{4} + \frac{1}{4} 5^{n+1} = \frac{1}{4}\left(5^{n+1} - 1\right)$$