Problem 1

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & \mathbb{R}^2 \arrow[r, "S"] & \mathbb{R}^5 \end{tikzcd}$$

(a) largest possible rank for $S\circ T$ is 2.

(b) $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad S\circ T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) the matrix T is 2×4 so there are at most 2 pivots, at least 2 free variables, so $\dim \mathrm{Nul}(T) \geq 2$.

If $T\vec{x} = \vec{0}$ then $(S\circ T)\vec{x} = S(T\vec{x}) = S(\vec{0}) = \vec{0}$. This proves that $\mathrm{Nul}(T) \subset \mathrm{Nul}(S\circ T)$.

So $\dim \mathrm{Nul}(S\circ T) \geq 2$, also.

So $S\circ T$ has at least 2 free variables, and since it is 5×4 it has at most 2 pivots, so $\mathrm{rk}(S\circ T) \leq 2$.

Problem 2(a)

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}. \qquad \det(\lambda I - A) = \lambda^2 + 3 = (\lambda - \sqrt{3}i)(\lambda + \sqrt{3}i).$$

$\lambda = \sqrt{3}i$ $\quad \begin{pmatrix} 1-\sqrt{3}i & 2 \\ -2 & -1-\sqrt{3}i \end{pmatrix}$ has solution $\begin{pmatrix} 2 \\ -1+\sqrt{3}i \end{pmatrix}$ = eigenvector,

$\lambda = -\sqrt{3}i$ $\quad$ complex conjugat will do: $\begin{pmatrix} 2 \\ -1-\sqrt{3}i \end{pmatrix}$.

general solution of $\vec{x}'(t) = A\vec{x}(t)$ is

$$\vec{x}(t) = c_1 e^{\sqrt{3}it} \begin{pmatrix} 2 \\ -1+\sqrt{3}i \end{pmatrix} + c_2 e^{-\sqrt{3}it} \begin{pmatrix} 2 \\ -1-\sqrt{3}i \end{pmatrix}$$ where c_1, c_2 are <u>cplx</u> numbers.

take one of the cplx solutions: $e^{\sqrt{3}it} \begin{pmatrix} 2 \\ -1+\sqrt{3}i \end{pmatrix} = (\cos\sqrt{3}t + i\sin\sqrt{3}t) \begin{pmatrix} 2 \\ -1+\sqrt{3}i \end{pmatrix}$

$$= \left((\cos\sqrt{3}t + i\sin\sqrt{3}t)\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right)\right) = \left[\cos\sqrt{3}t \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin\sqrt{3}t \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right] + i\left[\cos\sqrt{3}t \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} + \sin\sqrt{3}t \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]$$

and take real & imaginary part as basis for the real solutions:

$$\vec{x}(t) = a_1\left[\cos\sqrt{3}t \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin\sqrt{3}t \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}\right] + a_2\left[\cos\sqrt{3}t \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} + \sin\sqrt{3}t \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]$$ where a_1, a_2 are <u>real</u>.

Problem 2(b) $\quad A = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 1 & 1 \\ -1 & \lambda-2 & 1 \\ -1 & 1 & \lambda-2 \end{vmatrix} = (\lambda-3)^2(\lambda-2)$$

eigenvectors:

$\lambda=3$: $\quad A-3I = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}$ eigenvectors: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$\lambda=2$: $\quad A-2I = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

so A is diagonalizable, gen'l sol'n to $\vec{x}' = Ax$ is:

$$\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{x}(t) = e^{3t}\begin{pmatrix} c_1+c_2 \\ c_1 \\ c_2 \end{pmatrix} + e^{2t}\begin{pmatrix} c_3 \\ c_3 \\ c_3 \end{pmatrix}.$$

Problem 2(c)

$$A = \begin{pmatrix} 2 & 1 & & \\ & 2 & 2 & \\ & & 2 & 3 \\ & & & 2 \end{pmatrix}.$$

only eigenvalue is 2, with algebraic multiplicity 4.

Use formula:

$$\vec{x}(t) = e^{2t} \sum_{j=0}^{3} \frac{t^j}{j!}(A-2I)^j \vec{a}$$

where $\vec{a}$ is the gen'l solution to $(A-2I)^4 \vec{a} = \vec{0}$.

$$A-2I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 2 & 0 \\ & & 0 & 3 \\ & & & 0 \end{pmatrix} \qquad (A-2I)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ & 0 & 0 & 6 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \qquad (A-2I)^3 = \begin{pmatrix} 0 & 0 & 0 & 6 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \qquad (A-2I)^4 = (0).$$

So $\vec{a}$ is completely general, $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$.

$$(A-2I)^0 \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \qquad (A-2I)^1 \vec{a} = \begin{pmatrix} a_2 \\ 2a_3 \\ 3a_4 \\ 0 \end{pmatrix} \qquad (A-2I)^2 \vec{a} = \begin{pmatrix} 2a_3 \\ 6a_4 \\ 0 \\ 0 \end{pmatrix} \qquad (A-2I)^3 \vec{a} = \begin{pmatrix} 6a_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{x}(t) = e^{2t}\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + t\begin{pmatrix} a_2 \\ 2a_3 \\ 3a_4 \\ 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 2a_3 \\ 6a_4 \\ 0 \\ 0 \end{pmatrix} + \frac{t^3}{6}\begin{pmatrix} 6a_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

Problem 2(d).

Consider the 2x2 complex matrix $T = \begin{pmatrix} 2+3i & 1 \\ 0 & 2+3i \end{pmatrix}$.

$T\begin{pmatrix}1\\0\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}i\\0\end{pmatrix}$

$T\begin{pmatrix}i\\0\end{pmatrix} = -3\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}i\\0\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2+3i\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\end{pmatrix} + 3\begin{pmatrix}0\\i\end{pmatrix}$

$T\begin{pmatrix}0\\i\end{pmatrix} = \begin{pmatrix}i\\2i-3\end{pmatrix} = \begin{pmatrix}i\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\end{pmatrix} + 2\begin{pmatrix}0\\i\end{pmatrix}$

T is a linear transformation $T:\mathbb{C}^2 \to \mathbb{C}^2$ but we can identify $\mathbb{C}^2 = \mathbb{R}^4$ using the basis $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}i\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\i\end{pmatrix}$ which is a real basis for $\mathbb{C}^2$.

In this basis T has matrix $\begin{pmatrix} 2 & -3 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 3 & 2 \end{pmatrix}$ which is the given matrix.

So we can identify the real transformation $A: \mathbb{R}^4 \to \mathbb{R}^4$ with the complex transformation $T: \mathbb{C}^2 \to \mathbb{C}^2$.

T is not diagonalizable, but there is only one eigenvalue $\lambda = 2+3i$, of alg mult 2.

$(T-(2+3i)I)^2 = \begin{pmatrix}0&1\\0&0\end{pmatrix}^2 = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ so the gen'l solution to $(T-\lambda I)^2 \vec{a} = \vec{0}$ is $\vec{a}$ arbitrary. So the gen'l solution to $\vec{x}'(t) = T\vec{x}(t)$ is

$$\vec{x}(t) = e^{(2+3i)t}\left(\begin{pmatrix}a_1\\a_2\end{pmatrix} + t\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}\right) = e^{2+3it}\begin{pmatrix}a_1+ta_2\\a_2\end{pmatrix}$$

remember that a_1, a_2 are cplx: $a_1 = b_1 + ic_1$ $\quad a_2 = b_2 + ic_2$ with $b_1 b_2\ c_1\ c_2$ real.

$$\vec{x}(t) = e^{(2+3i)t}\begin{pmatrix}b_1 + ic_1 + b_2 t + ic_2 t\\ b_2 + ic_2\end{pmatrix}$$

2(d) continued:

$$\vec{x}(t) = e^{2t}\left(\cos 3t + i\sin 3t\right)\begin{pmatrix} b_1 + b_2 t + i(c_1 + c_2 t) \\ b_2 + i c_2 \end{pmatrix}$$

$$= e^{2t}\cos 3t\begin{pmatrix} b_1 + b_2 t \\ b_2 \end{pmatrix} - e^{2t}\sin 3t\begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} + i e^{2t}\sin 3t\begin{pmatrix} b_1 + b_2 t \\ b_2 \end{pmatrix} + i e^{2t}\cos 3t\begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix}$$

$$= e^{2t}\left(\cos 3t\,(b_1 + b_2 t) - \sin 3t\,(c_1 + c_2 t)\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ e^{2t}\left(\sin 3t\,(b_1 + b_2 t) + \cos 3t\,(c_1 + c_2 t)\right)\begin{pmatrix} i \\ 0 \end{pmatrix}$$

$$+ e^{2t}\left(b_2\cos 3t - c_2\sin 3t\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

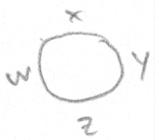
$$+ e^{2t}\left(b_2\sin 3t + c_2\cos 3t\right)\begin{pmatrix} 0 \\ i \end{pmatrix}$$

we write this in $\mathbb{R}^4$ instead of $\mathbb{C}^2$:

$$\vec{x}(t) = e^{2t}\begin{pmatrix} (b_1 + b_2 t)\cos 3t - (c_1 + c_2 t)\sin 3t \\ (b_1 + b_2 t)\sin 3t + (c_1 + c_2 t)\cos 3t \\ b_2\cos 3t - c_2\sin 3t \\ b_2\sin 3t + c_2\cos 3t \end{pmatrix}$$

$$= b_1 e^{2t}\begin{pmatrix} \cos 3t \\ \sin 3t \\ 0 \\ 0 \end{pmatrix} + b_2 e^{2t}\begin{pmatrix} t\cos 3t \\ t\sin 3t \\ \cos 3t \\ \sin 3t \end{pmatrix} + c_1 e^{2t}\begin{pmatrix} -\sin 3t \\ \cos 3t \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -t\sin 3t \\ t\cos 3t \\ -\sin 3t \\ \cos 3t \end{pmatrix}$$

Problem 3



$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} \frac12 y_n + \frac12 w_n \\ \frac12 x_n + \frac12 z_n \\ \frac12 w_n + \frac12 y_n \\ \frac12 x_n + \frac12 z_n \end{pmatrix} = \begin{pmatrix} 0 & \frac12 & 0 & \frac12 \\ \frac12 & 0 & \frac12 & 0 \\ 0 & \frac12 & 0 & \frac12 \\ \frac12 & 0 & \frac12 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix}$$

transition matrix $A = \begin{pmatrix} 0 & \frac12 & 0 & \frac12 \\ \frac12 & 0 & \frac12 & 0 \\ 0 & \frac12 & 0 & \frac12 \\ \frac12 & 0 & \frac12 & 0 \end{pmatrix}$

eigenvalues: $\det(\lambda I - A) = \begin{vmatrix} \lambda & -\frac12 & 0 & -\frac12 \\ -\frac12 & \lambda & -\frac12 & 0 \\ 0 & -\frac12 & \lambda & -\frac12 \\ -\frac12 & 0 & -\frac12 & \lambda \end{vmatrix} \begin{matrix} \\ \\ \\ -\mathrm{II} \end{matrix} = \begin{vmatrix} \lambda & -\frac12 & 0 & -\frac12 \\ -\frac12 & \lambda & -\frac12 & 0 \\ 0 & -\frac12 & \lambda & -\frac12 \\ 0 & -\lambda & 0 & \lambda \end{vmatrix}$

$$= -\lambda \begin{vmatrix} \lambda & 0 & -\frac12 \\ -\frac12 & -\frac12 & 0 \\ 0 & \lambda & -\frac12 \end{vmatrix} + \lambda \begin{vmatrix} \lambda & -\frac12 & 0 \\ -\frac12 & \lambda & -\frac12 \\ 0 & -\frac12 & \lambda \end{vmatrix}$$

$$= -\lambda^2 \begin{vmatrix} -\frac12 & 0 \\ \lambda & -\frac12 \end{vmatrix} + \frac12\lambda \begin{vmatrix} -\frac12 & -\frac12 \\ 0 & \lambda \end{vmatrix} + \lambda^2 \begin{vmatrix} \lambda & -\frac12 \\ -\frac12 & \lambda \end{vmatrix} + \frac12\lambda \begin{vmatrix} -\frac12 & -\frac12 \\ 0 & \lambda \end{vmatrix}$$

$$= -\frac14\lambda^2 - \frac14\lambda^2 + \lambda^4 - \frac14\lambda^2 - \frac14\lambda^2 = \lambda^4 - \lambda^2 = \lambda^2(\lambda^2 - 1) = \lambda^2(\lambda+1)(\lambda-1)$$

<u>eigenvalues</u> are $\lambda = 0$, $\lambda = 1$, $\lambda = -1$.

<u>eigenvectors</u>:

$\lambda = 0$: $(A - 0I) = A \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ gives $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1$: $(A - I) = \begin{pmatrix} -1 & \frac12 & 0 & \frac12 \\ \frac12 & -1 & \frac12 & 0 \\ 0 & \frac12 & -1 & \frac12 \\ \frac12 & 0 & \frac12 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ gives $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = -1$: $(A + I) = \begin{pmatrix} 1 & \frac12 & 0 & \frac12 \\ \frac12 & 1 & \frac12 & 0 \\ 0 & \frac12 & 1 & \frac12 \\ \frac12 & 0 & \frac12 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ gives $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

So A is diagonalizable, let's compute A^n.

Change of basis matrix is $P = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

Compute the inverse:

$$\left(\begin{array}{cccc|cccc} -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 2 & 2 & 0 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccccccc} & & \dots & & & \dots & & \\ & & & & & & & \\ & & & & & & & \\ 0 & 0 & 0 & 4 & -1 & 1 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & -1/4 & 1/4 & -1/4 & 1/4 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & -3/4 & -1/4 & 1/4 & -1/4 \\ 0 & 1 & -1 & 0 & -1/4 & -3/4 & -1/4 & 1/4 \\ 0 & 0 & 1 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & -1/4 & 1/4 & -1/4 & 1/4 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2/4 & 0 & 2/4 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2/4 & 0 & 2/4 \\ 0 & 0 & 1 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & -1/4 & 1/4 & -1/4 & 1/4 \end{array}\right) \qquad P^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

so $A = PDP^{-1}$ and $A^n = PD^nP^{-1}$

$$A^n = \frac{1}{4}\begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0^n & & & \\ & 0^n & & \\ & & 1^n & \\ & & & (-1)^n \end{pmatrix}\begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

if $n > 0$ then $0^n = 0$ so:

$$= \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 & -(-1)^n \\ 0 & 0 & 1 & (-1)^n \\ 0 & 0 & 1 & -(-1)^n \\ 0 & 0 & 1 & (-1)^n \end{pmatrix}\begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+(-1)^n & 1-(-1)^n & 1+(-1)^n & 1-(-1)^n \\ 1-(-1)^n & 1+(-1)^n & 1-(-1)^n & 1+(-1)^n \\ 1+(-1)^n & 1-(-1)^n & 1+(-1)^n & 1-(-1)^n \\ 1-(-1)^n & 1+(-1)^n & 1-(-1)^n & 1+(-1)^n \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix} \text{ if } n \text{ is even} \qquad = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \text{ if } n \text{ is odd}.$$

So after 10 sounds of the gong

$$x_{10} = \tfrac{1}{2} x_0 + \tfrac{1}{2} z_0$$

$$y_{10} = \tfrac{1}{2} y_0 + \tfrac{1}{2} w_0$$

$$z_{10} = \tfrac{1}{2} x_0 + \tfrac{1}{2} z_0$$

$$w_{10} = \tfrac{1}{2} y_0 + \tfrac{1}{2} w_0$$

so after 10 sounds of the gong each knight has ½ of what he started out with plus half of what the knight vis à vis started out with.

after 15 sounds of the gong

$$x_{15} = \tfrac{1}{2} y_0 + \tfrac{1}{2} w_0$$

$$y_{15} = \tfrac{1}{2} x_0 + \tfrac{1}{2} z_0$$

$$z_{15} = \tfrac{1}{2} y_0 + \tfrac{1}{2} w_0$$

$$w_{15} = \tfrac{1}{2} x_0 + \tfrac{1}{2} z_0$$

after 15 sounds of the gong each knight has ½ of what his two neighbours started out with.

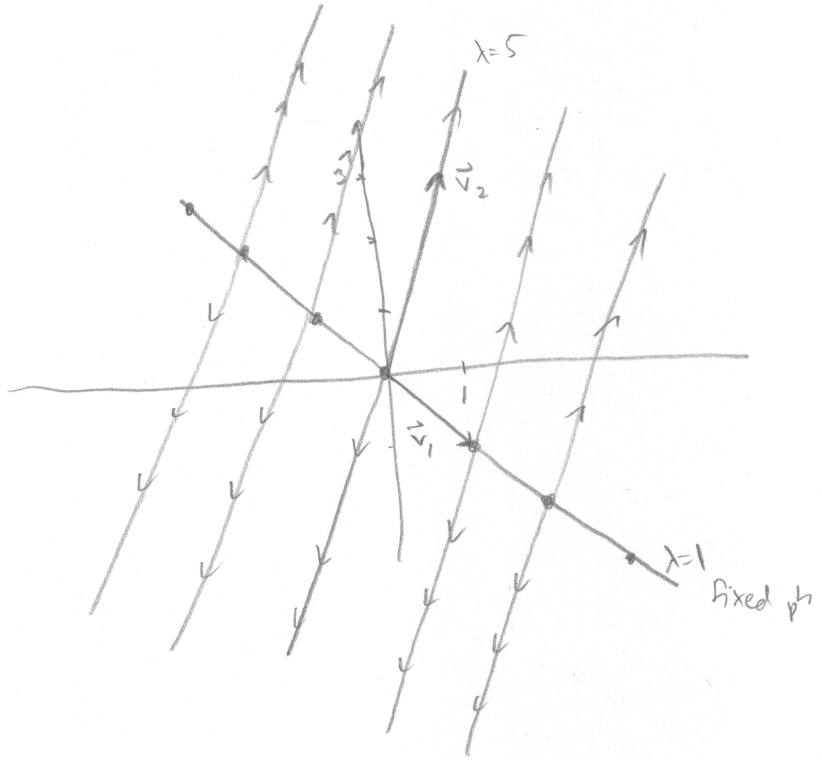
(the amount of cereal in the bowls keeps oscillating between these 2 distributions.)

Problem 4 (a)

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \qquad \det(\lambda I - A) = \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5)$$

$\lambda = 1$ $\quad A - I = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{v}_1$

$\lambda = 5$ $\quad A - 5I = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \to \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}$ eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \vec{v}_2$



the trajectories are straight lines parallel to the line spanned by $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

the phase vector is being "repelled" by the line of fixed pts spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Problem 4 (b)

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \qquad \det(\lambda I - A) = \lambda^2 - 2\lambda + 3$$

with roots $\lambda = 1 \pm \sqrt{2}\, i$

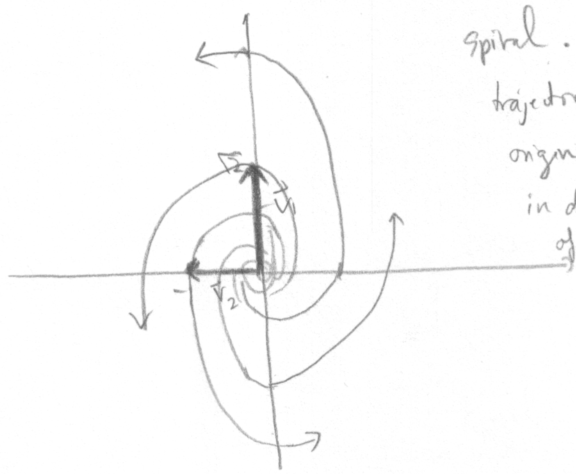
use eigenvalue $\lambda = 1 + \sqrt{2}\, i$ with absolute value $r = \sqrt{3}$
and argument $\Theta = \arctan \sqrt{2}$, $45^\circ < \Theta < 60^\circ$.

corresponding eigenvector:

$$A - \lambda I = \begin{pmatrix} -\sqrt{2}\, i & -1 \\ 2 & -\sqrt{2}\, i \end{pmatrix} \text{ which has solution } \begin{pmatrix} -1 \\ \sqrt{2}\, i \end{pmatrix}$$

as real basis use $\operatorname{im}\begin{pmatrix} -1 \\ \sqrt{2}\, i \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = \vec{v}_1$ and $\operatorname{Re}\begin{pmatrix} -1 \\ \sqrt{2}\, i \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \vec{v}_2$.

in this basis, A looks like multiplication by the cplx number $1 + \sqrt{2}\, i$.



spiral.
trajectories go counterclockwise away from origin. trajectories are closest together in direction of $\vec{v}_2$, farthest apart in direction of $\vec{v}_1$

Problem 5. the quadric is $Q(x,y,z)=1$ for the quadratic form

$$Q(x,y,z) = 4x^2+4y^2+4z^2+2xy+2xz+2yz$$

with matrix $A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

the principal axes are the eigenspaces of this symmetric matrix.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & -1 & -1 \\ -1 & \lambda-4 & -1 \\ -1 & -1 & \lambda-4 \end{vmatrix} = (\lambda-6)(\lambda-3)^2$$

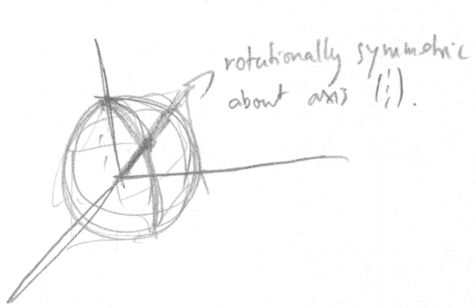
$\underline{\underline{\lambda=6}}$ $\quad A-6I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ eigenspace $= \operatorname{span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

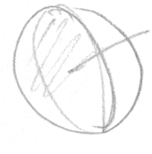
$\underline{\underline{\lambda=3}}$ $\quad A-3I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ eigenspace $= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^{\perp}$.

this eigenspace has dimension 2.
all lines in this plane are principal axes.

in coords given by principal axes, the quadratic form is $Q(x',y',z') = 6x'^2 + 3y'^2 + 3z'^2$

this has intersection pts with all principal axes, so it is an <u>ellipsoid</u>.





Problem 6.

We are asked to find an orthogonal basis for Q.

matrix of Q is $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{pmatrix} = A$.

$Q\begin{pmatrix}1\\0\\0\end{pmatrix} = (1\,0\,0)A\begin{pmatrix}1\\0\\0\end{pmatrix} = 3$ so take $\begin{pmatrix}1\\0\\0\end{pmatrix}$ as first basis vector.

the 2^{nd} basis vector $\vec{x}$ has to be orth to the first, i.e.

$Q\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{x}\right) = 0$ or $(1\,0\,0)A\vec{x} = 0$ or $(3\,1\,1)\vec{x} = 0$

For example, $\vec{x} = \begin{pmatrix}-1\\3\\0\end{pmatrix}$.

the 3^{rd} basis vector $\vec{x}$ has to be orth to the 1^{st} two, so

$\begin{pmatrix}1 & 0 & 0\\ -1 & 3 & 0\end{pmatrix} A\vec{x} = \vec{0}$ or $\begin{pmatrix}1 & 0 & 0\\ -1 & 3 & 0\end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 11 & 2\end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & 2/11 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3/11 \\ 0 & 1 & 2/11 \end{pmatrix}$ has solution $\begin{pmatrix}-3\\-2\\11\end{pmatrix}$.

So the substitution $\vec{x} = \begin{pmatrix} 1 & -1 & -3 \\ 0 & 3 & -2 \\ 0 & 0 & 11 \end{pmatrix}\vec{x}'$ will get rid of the cross terms.

Check:

$$\begin{aligned} Q(x', y', z') = {} & 3(x'-y'-3z')^2 + 4(3y'-2z')^2 + 5(11z')^2 + 2(x'-y'-3z')(3y'-2z') \\ & + 2(x'-y'-3z')(11z') \\ & + 2(3y'-2z)(11z') \end{aligned}$$

$$= 3x'^2 + 33y'^2 + 550z'^2.$$