Final Exam

Thursday, April 18, 2019

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (2+2+3+3 points)
Let V be a vector space over the field $\mathbb{F}$, and $f : V \to V$ an endomorphism.

(a) Define what an eigenvalue of f is.

(b) Define what an eigenvector of f is.

(c) Suppose that $v_1, \ldots, v_k$ are eigenvectors of f, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that if $\lambda_i = \lambda_j$ only if $i = j$, then $v_1, \ldots, v_k$ is a linearly independent family of vectors in V.

(d) Suppose A is a real symmetric matrix. Prove that eigenvectors corresponding to different eigenvalues are orthogonal to each other.

Problem 2. (3+3+3 points)
The theorem of Cayley-Hamilton says that if you formally substitute a matrix into its own characteristic polynomial, you obtain the zero matrix. For example, if the characteristic polynomial of the matrix A is $\lambda^2 - 2\lambda + 2$, then $A^2 - 2A + 2E = 0$, where E is the 2×2-identity matrix. (Note that the constant term of the polynomial is $2\lambda^0$, which you need to replace by $2A^0 = 2E$.)

(a) Verify the theorem of Cayley-Hamilton for the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(b) Verify the theorem of Cayley-Hamilton for the shear matrix

$$\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$$

(c) Prove the theorem of Cayley-Hamilton for diagonalizable matrices.

Problem 3. (4+4 points)

(a) Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for the real symmetric matrix

$$A = \begin{pmatrix} 5 & \sqrt{3} & -\sqrt{2} \\ \sqrt{3} & 3 & \sqrt{6} \\ -\sqrt{2} & \sqrt{6} & 4 \end{pmatrix}$$

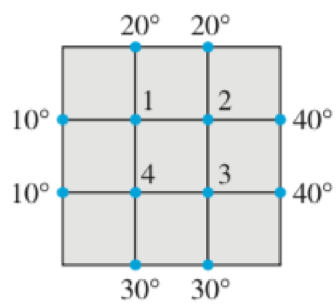
(b) Find a formula for A^n.

Problem 4. (10 points)
Suppose

$$a_{n+1} = 5a_n - 6b_n\,, \qquad b_{n+1} = 3a_n - 4b_n\,, \qquad a_0 = 1\,, \qquad b_0 = 1\,.$$

Find formulas for a_n, b_n.

Problem 5. (6 points)
The picture below represents a mesh of wires. At each of the 4 nodes labelled $1, \ldots, 4$, the temperature is the average of the temperatures at the four nearby nodes. Find the temperature at each of the nodes.



Problem 6. (6 points)
Compute the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 & 3 \\ 2 & 0 & 2 & x & 0 \end{pmatrix}$$

Problem 7. (8 points)
The augmented coefficient matrix of an inhomogeneous linear system of equations is

$$\begin{pmatrix} 0 & 2-t & 1 & -3t & | & t-2t \\ 0 & 1 & 0 & 3 & | & 2 \\ 0 & 1 & 0 & 4+t & | & 2+t \\ 0 & 2 & 1 & 0 & | & 2t \end{pmatrix}$$

(a) under what conditions on t are there no solution, a unique solution, or infinitely many solutions?
(b) Find the dimension of the solution space of the associated homogeneous system.
(c) Find the rank of the given augmented matrix.

Problem 8. (12 points)
True or false? Write the answer in your booklet. (No justifications necessary.)

(a) Every real symmetric matrix with only one eigenvalue is a diagonal matrix.

(b) Let $f : V \to V$ be a self-adjoint endomorphism of a Euclidean vector space V. Every basis of V consisting of eigenvectors for f is an orthogonal system.

(c) If A and B are similar matrices, i.e., if there exists an invertible matrix P, such that $A = PBP^{-1}$, then A and B have the same eigenvectors.

(d) If λ is a non-zero eigenvalue of an automorphism f, then $1/\lambda$ is an eigenvalue of f^{-1}.

(e) If f is an endomorphism of a finite-dimensional vector space V and $f(-v) = \lambda v$, for some scalar λ, and some non-zero vector $v \in V$, then v is an eigenvector of f.

(f) Every orthogonal endomorphism of a Euclidean vector space has determinant 1.

(g) For every subspace $U \subset V$ of a Euclidean vector space, the orthogonal projection onto U is a self-adjoint endomorphism of V.

(h) Let $A \in M(n \times n, \mathbb{F})$, and $\det A = 0$. Then $Ax = b$ is solvable only for some b (not all b), and never uniquely.

(i) If b is one of the columns of A, then $Ax = b$ is solvable.

(j) If $U_1, U_2 \subset V$ are subspaces of the vector space V, then $\dim(U_1 \cap U_2) \geq \dim U_1 + \dim U_2 - \dim V$.

(k) If $f : V \to W$ has rank 3, and $g : W \to U$ has rank 4, then the rank of $g \circ f$ is at least 3.

(l) If A is a 3×3-matrix, then $\det(3A) = 3 \det(A)$.