Final Exam

8:30–11:00, Monday, December 17, 2018

No books. No notes. No calculators. No electronic devices of any kind.

Problem 1. (3+3 points)

(a) List the conditions a matrix has to satisfy to be in *reduced row echelon form*.

(b) Group the following matrices together according to row equivalence:

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$$

Problem 2. (5 points)

Find a formula for A^n, $n \geq 1$, where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} .$$

Problem 3. (3+3 points)

(a) Define the term *Sylvester basis.* Carefully state the context.

(b) Find a Sylvester basis for the quadratic form

$$q(x, y, z) = x^2 + 2xz - 3yz \,.$$

Problem 4. (2+4 points)

Let $A \in M(3 \times 3, \mathbb{R})$ be a matrix which is both symmetric and orthogonal.

(a) Prove that $A^2 = I_3$, the identity matrix.

(b) Prove that $A = \pm I_3$, or A is a reflection across a plane through the origin, or a 180° degree rotation about a line through the origin.

Problem 5. (2+2+3 points)

(a) Define the term *similarity*, as it applies to matrices.

(b) Give 3 matrices $A, B, C \in M(3 \times 3, \mathbb{R})$, which are mutually not similar.

(c) Prove that the two real matrices ($n \geq 1$)

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 2 & 3 & \dots & n \\ 0 & 0 & 3 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 0 & \dots & 0 \\ 1 & 2 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$$

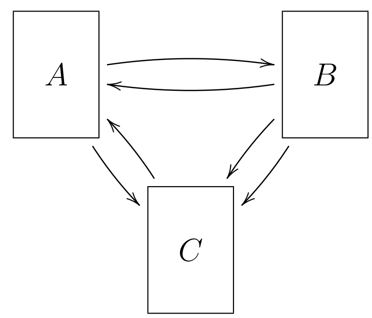
are similar.

Problem 6. (5 points)

Let V be a real vector space of dimension 2, and let $f : V \to V$ be an endomorphism of V. Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of f, that λ is the only eigenvalue of f, and that f is not diagonalizable. Prove that there exists a basis $\mathcal{B}$ of V, such that

$$[f]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} .$$

Problem 7. (4+1 points)
Below is a simplified model of the internet. There are 3 webpages, which are hyperlinked as in the diagram:



In words:

Page A has one link to Page B and one link to Page C,
Page B has two links to Page C and one link to Page A,
Page C has one link to Page A.

Assume there are 1 million users, and that every user always spends 1 minute on each webpage and then clicks on a random link. At time zero, each user starts on a random page. After a short time, the internet will reach equilibrium, which means that the number of users on each page will be (approximately) constant from one minute to the next.

(a) Find which proportion of users are on each of the three pages in this equilibrium state.

The fractions you obtain for the webpages A, B and C are (up to logarithmic scaling) the *Page Ranks* of the three web pages, named after Larry Page, co-founder of Google. The Page Rank is an important ingredient in Google's ranking of pages, and contributed much to the early success and therefore current dominance of Google.

(b) Rank the three web pages by Page Rank.

Problem 8. (10 points)
True or False? **Give the answers in your exam booklet**.

(a) the map $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = x + y + 1$ is linear.

(b) the map $f : \mathbb{C} \to \mathbb{C}$, defined by $f(x + iy) = x - iy$ is $\mathbb{C}$-linear.

(c) Every family of 4 vectors in $\mathbb{F}^3$ is linearly dependent.

(d) Every family of 3 vectors in $\mathbb{F}^4$ is linearly independent.

(e) Every linear system of 6 equations with 8 indeterminants admits a solution.

(f) If two matrices have the same rank, then they are row equivalent.

(g) If $U, W \subset V$ are subspaces of the finite-dimensional vector space V, then $\dim(U + W) = \dim U + \dim W$.

(h) If $U, W \subset V$ are subspaces of the finite-dimensional vector space V, and $\dim(U + W) = \dim U + \dim W$, then $U \cap W = \{0\}$.

(i) If two square matrices have the same characteristic polynomial, then they have the same rank.

(j) If two square matrices have the same characteristic polynomial, they are similar.

(k) If, for every eigenvalue of a matrix A, the algebraic multiplicity is equal to the geometric multiplicity, then A is diagonalizable.

(l) The matrix of a reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ is necessarily symmetric.

(m) If two real symmetric matrices have the same characteristic polynomial, they commute with each other.

(n) If a self-adjoint linear map $f : V \to V$ satisfies $f(U) \subset U$, for a subspace $U \subset V$, then $f(U^\perp) \subset U^\perp$.

(o) Every Sylvester basis of a quadratic form consists of eigenvectors for the symmetric matrix associated to the quadratic form.

(p) If a matrix $A \in M(2 \times 2, \mathbb{C})$ satisfies $A^3 = I_2$, then A is diagonalizable.

(q) Every orthogonal matrix has determinant 1.

(r) If $f : V \to V$ is a self-adjoint endomorphism of a Euclidean vector space V, then the matrix of f with respect to any basis of V is symmetric.

(s) For every matrix $A \in SO(2)$, there exists a matrix B, such that $B^2 = A$.

(t) If the characteristic polynomial of an $n \times n$ matrix A is t^n, then $A^n = 0$.