# A few challenge problems

**Problem 1.** (0 points)
Let $q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, with associated matrix $A \in \mathrm{Sym}(n \times n, \mathbb{R})$. Prove that $x \in \mathbb{R}^n$ is an eigenvector for $A$ if and only if $x$ is a critical point for the distance from the origin on the hypersurface $q(x) = 1$. Muse about the connection between eigenvalues and Lagrange multipliers.

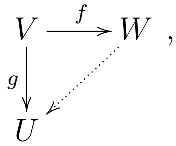
**Problem 2.** (0 points)
Let $R \in SO(3)$. Prove that $R$ is a rotation, as follows.

By considering the characteristic polynomial of $R$, and what we know about its constant term, use the intermediate value theorem to prove that 1 is a root of the characteristic polynomial, and hence $R$ has a line $L$ of fixed points. Prove that $R(L^\perp) = L^\perp$, and hence that $R$ restricted to $L^\perp$ is a 2-dimensional rotation. This implies that $R$ is a rotation about $L$.

**Problem 3.** (0 points)
Prove the SAS congruence theorem: let $X$ be a Euclidean plane, i.e., an affine space for a 2-dimensional Euclidean vector space. Call two triangles (i.e., non-collinear triples of points) *congruent*, if there exists an isometry $\phi : X \to X$ mapping one triangle onto the other. Prove that if $d(A, B) = d(A', B')$, $d(A, C) = d(A', C')$ and $\measuredangle(BAC) = \measuredangle(B'A'C')$, then the triangles $(A, B, C)$ and $(A', B', C')$ are congruent.

**Problem 4.** (0 points)
Let $X$ be a Euclidean plane associated with the 2-dimensional Euclidean vector space $E$. Prove that every isometry $\phi : X \to X$ is either a translation, a rotation, or a glide reflection. A *rotation* of $X$ is an isometry $\phi : X \to X$, such that there exists a point $P_0$, and an angle $\theta \in (0, \pi]$, such that for all $P \in X$, $P \neq P_0$, we have $\measuredangle(PP_0\phi(P)) = \theta$. A *glide reflection* of $X$ is an isometry $\phi : X \to X$, such that there exists a line $L \subset X$, such that $\phi$ is the composition of a translation parallel to $L$, and a reflection across $L$.

**Problem 5.** (0 points)
Let $f : V \to W$ be an epimorphism of $\mathbb{F}$-vector spaces, and $g : V \to U$ an arbitrary homomorphispm of $\mathbb{F}$-vector spaces. Prove that the dotted arrow exists




making the diagram commute, if and only if $g(\ker f) = \{0\}$.

**Problem 6.** (0 points)
Let $V$ be a vector space over the field $\mathbb{F}$. A *linear form* on $V$ is a linear map $\psi : V \to \mathbb{F}$. The vector space of linear forms $\operatorname{Hom}(V, \mathbb{F})$ is called the *dual space* of $V$, denoted

$$V^* = \operatorname{Hom}(V, \mathbb{F}).$$

(a) If $\dim V = n < \infty$, let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis of $V$. Define linear forms $(v_1^*, \ldots, v_n^*)$ by the formula

$$v_i^*(v_j) = \delta_{ij}.$$

Prove that $\mathcal{B}^* = (v_1^*, \ldots, v_n^*)$ is a basis of $V^*$; it is called the *dual basis* of $\mathcal{B}$. Deduce that if $V$ is finite-dimensional, $\dim V = \dim V^*$. (If $V = \mathbb{F}^n$, note that $V$ is the space of column vectors of length $n$, and $V^*$ is the space of row vectors of length $n$, because $V^*$ is the space of $1 \times n$-matrices.)

(b) Let $f : V \to W$ be a linear map. We define the *dual of* $f$ to be the linear map

$$\begin{aligned} f^* : W^* &\longrightarrow V^* \\ \psi &\longmapsto \psi \circ f. \end{aligned}$$

Prove that $f^*$ is, indeed, linear. Prove that if $\mathcal{B}$ is a basis of $V$ and $\mathcal{C}$ is a basis of $W$, then

$$[f^*]_{\mathcal{B}^*}^{\mathcal{C}^*} = ([f]_{\mathcal{C}}^{\mathcal{B}})^t.$$

Prove that if $V \to W$ is an epimorphism, then $W^* \to V^*$ is a monomorphism. Prove that if $V$ is finite-dimesional, and if $W \to V$ is a monomorphism, then $V^* \to W^*$ is an epimorphism.

(c) For any vector space $V$ over $\mathbb{F}$, we have a *canonical* homomorphism (ev stands for 'evaluation homomorphism')

$$\begin{aligned} \operatorname{ev} : V &\longrightarrow V^{**} \\ v &\longmapsto [\psi \mapsto \psi(v)]. \end{aligned}$$

Prove that ev is injective. Prove that ev is an isomorphism, if $V$ is finite-dimensional. (This is false if $V$ is infinite-dimensional.) Deduce that if $V$ and $W$ are finite-dimensional, and $f : V \to W$ is a homomorphism, then $\operatorname{rk} f = \operatorname{rk} f^{**}$.

(d) For a subspace $W \subset V$, we define

$$W^\perp = \ker(V^* \to W^*) = \{\psi \in V^* \mid \forall w \in W : \psi(w) = 0\}.$$

Prove that if $V$ is finite-dimensional we have $\dim W^\perp = \dim V - \dim W$.

(e) Prove that for every homomorphism $f : V \to U$, we have $\operatorname{im}(f^*) \subset (\ker f)^\perp$. Deduce that if $V$ is finite-dimensional, we have $\operatorname{rk} f^* \leq \operatorname{rk} f$.

(f) Let $f : V \to U$ be a homomorphism of finite-dimensional $\mathbb{F}$-vector spaces. Prove that $\operatorname{rk} f^* = \operatorname{rk} f$, by using the double dual. (This gives a new proof of the equality of row and column rank.)

(g) Deduce that for every homomorphism of finite-dimensional vector spaces we have

$$\boxed{\mathrm{im}(f^*) = (\ker f)^\perp}.$$