Dynamical Systems and Matrix Algebra

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Abstract

This is a review of how matrix algebra applies to linear dynamical systems. We treat the discrete and the continuous case.

Contents

Introduction

This notes are in somewhat preliminary form. The content is not yet stable. Watch for updates.

1 Discrete Dynamical Systems

1.1 A Markov Process

A migration example

Let us start with an example. Consider the populations of the two cities Vancouver and Richmond. The following graphic shows the yearly migration patterns.

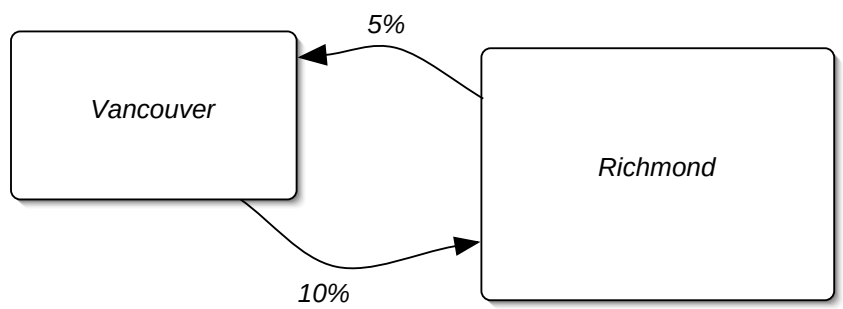


Figure 1: Yearly migration patterns between Vancouver and Richmond

This means that every year, 5% of the population of Richmond moves to Vancouver and 10% of the Vancouver population moves to Richmond. We neglect all other effects on the populations of the two cities: no one gets born or dies, no one moves elsewhere, no one moves from outside Vancouver or Richmond to one of the two cities. Let us measure population size in multiple of 1,000.

Translating the problem into matrix algebra

To describe the populations, we will use *population vectors*. These will be vectors $\binom{v}{r}$, where v is the Vancouver population and r the Richmond population. Thus a population vector of $\binom{300}{100}$ represents a population of 300,000 in Vancouver and 100,000 in Richmond.

If the population vector in the current year, 2007, is $\binom{300}{100}$, then next year the population vector will be $\binom{275}{125}$. This is because 30 kilo-people (10% of 300 kilo-people) move from Vancouver to Richmond and 5 kilo-people (5% of 100 kilo-people) move from Richmond to Vancouver, resulting in a net gain of 25 kilo-people for Richmond.

Translating Figure 1 into equations, we arrive at

$$\begin{aligned} v_{2008} &= .9\, v_{2007} + .05\, r_{2007} \\ r_{2008} &= .1\, v_{2007} + .95\, r_{2007} \end{aligned}$$

To save space, let us write the year 2007 as n and the year 2008 as $n+1$. Then we can rewrite these equations as

$$\begin{aligned} v_{n+1} &= .9\, v_n + .05\, r_n \\ r_{n+1} &= .1\, v_n + .95\, r_n \end{aligned}$$

because the Vancouver population next year consists of 90% of the Vancouver population this year and 5% of the Richmond population this year. Similarly, the Richmond population next year consists of 95% of the Richmond population this year plus 10% of the Vancouver population this year.

In vector form:

$$\begin{pmatrix} v_{n+1} \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} .9\, v_n + .05\, r_n \\ .1\, v_n + .95\, r_n \end{pmatrix} = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v_n \\ r_n \end{pmatrix}$$

Thus we see that the change in the population vector from one year to the next is given by a matrix transformation:

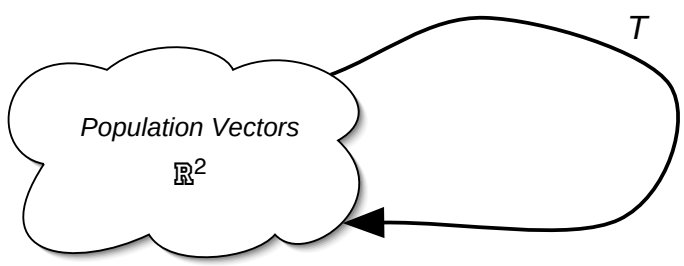


Figure 2: The matrix transformation T takes population vectors as input and as output

The vector space of population vectors is $\mathbb{R}^2$. Of course, many vectors in $\mathbb{R}^2$ do not make sense as population vectors, such as vectors with negative or irrational components. But it is still better to work with $\mathbb{R}^2$, rather than the set of sensible population vectors, because we can freely take linear combinations in $\mathbb{R}^2$.

The linear transformation is therefore

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

The input vector space and the output vectors space are both equal to $\mathbb{R}^2$. Let us write A for the matrix of T:

$$M(T) = A$$

We have seen that

$$A = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix}$$

We multiply population vectors by the matrix A to go from one year to the next.

$$\begin{pmatrix} v_{n+1} \\ r_{n+1} \end{pmatrix} = A \begin{pmatrix} v_n \\ r_n \end{pmatrix}$$

If we write $\vec{p} = \left(\begin{smallmatrix} v \\ r \end{smallmatrix}\right)$ we can write this even shorter as

$$\boxed{\boxed{\vec{p}_{n+1} = A\,\vec{p}_n}} \tag{1}$$

We call A the *transition matrix* of the dynamical system.

The main feature of such a dynamical system is that the input and output vectors are of the same type. In our case, they are both population vectors. So we can *iterate* the transformation. We can compute the population vector two years from now (just plug Equation (1) into itself):

$$\vec{p}_{n+2} = A\vec{p}_{n+1} = A(A\vec{p}_n) = A^2\vec{p}_n$$

Or 10 years from now:

$$\vec{p}_{n+10} = A^{10}\,\vec{p}_n$$

If we declare the current year to be year 0, then the population vector 10 years from now is

$$\vec{p}_{10} = A^{10}\,\vec{p}_0$$

With our above example:

$$\vec{p}_{10} = \begin{pmatrix} v_{10} \\ r_{10} \end{pmatrix} = A^{10} \begin{pmatrix} v_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix}^{10} \begin{pmatrix} 300 \\ 100 \end{pmatrix}$$

To compute the powers of a matrix is in general not so easy. Of course, we can just compute by brute force, but there is a better way, which gives us more insight: the method of *eigenvalues*.

Finding the equilibrium

Let us plot the population development for this example. Here are the first four values (they are rounded off to the nearest 1,000):

$$\begin{aligned}
\begin{pmatrix} v_0 \\ r_0 \end{pmatrix} &= \begin{pmatrix} 300 \\ 100 \end{pmatrix} \\
\begin{pmatrix} v_1 \\ r_1 \end{pmatrix} &= \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} 300 \\ 100 \end{pmatrix} = \begin{pmatrix} 275 \\ 125 \end{pmatrix} \\
\begin{pmatrix} v_2 \\ r_2 \end{pmatrix} &= \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} 275 \\ 125 \end{pmatrix} = \begin{pmatrix} 254 \\ 146 \end{pmatrix} \\
\begin{pmatrix} v_3 \\ r_3 \end{pmatrix} &= \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} 254 \\ 146 \end{pmatrix} = \begin{pmatrix} 236 \\ 164 \end{pmatrix}
\end{aligned}$$

We see that the Vancouver population decreases, as the Richmond population increases. Figure 3 shows a plot of the first 100 values of the population vector. It looks like the population vectors keep getting closer together. The more time goes by, the smaller is the difference between $\vec{p}_n$ and $\vec{p}_{n+1} = A\vec{p}_n$. If n gets very large, there is almost no difference between $\vec{p}_n$ and $A\vec{p}_n$. So it looks like the the population vector approaches a fixed point. As the years go by, we get closer and closer to a certain population vector $\vec{p}_\infty$ which satisfies the property

$$\boxed{\boxed{A\,\vec{p}_\infty = \vec{p}_\infty}} \tag{2}$$

To find the vector $\vec{p}_\infty = \left(\begin{smallmatrix} v \\ r \end{smallmatrix}\right)$, we rewrite this equation:

$$\begin{aligned}
\begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} &= \begin{pmatrix} v \\ r \end{pmatrix} \\
\begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix}
\end{aligned} \tag{3}$$

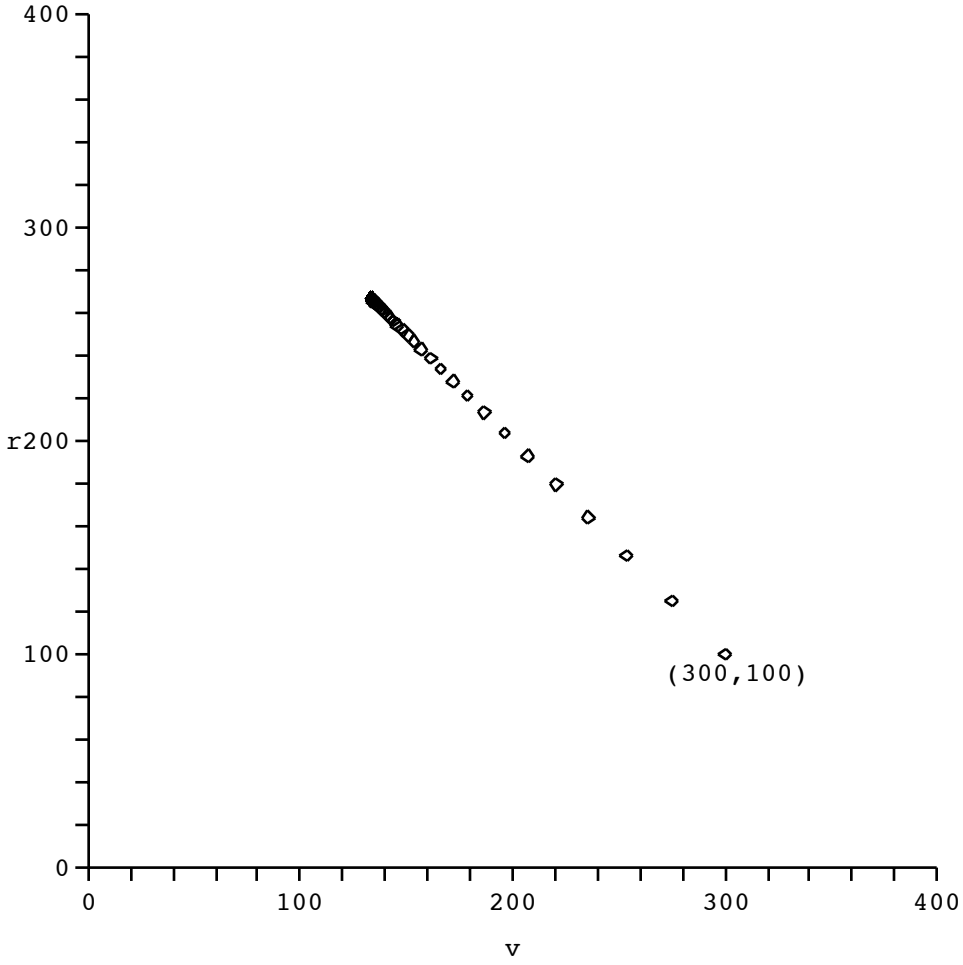


Figure 3: Phase portrait with initial condition $\binom{300}{100}$.

$$\begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} .9-1 & .05-0 \\ .1-0 & .95-1 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -.1 & .05 \\ .1 & -.05 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is a homogeneous system of equations for the indeterminants v and r, which we easily solve using Gaussian elimination:

$$\begin{pmatrix} -.1 & .05 \\ .1 & -.05 \end{pmatrix}_{R_1+R_2} \to \begin{pmatrix} -.1 & .05 \\ 0 & 0 \end{pmatrix}^{\cdot(-10)} \to \begin{pmatrix} 1 & -.5 \\ 0 & 0 \end{pmatrix}$$

We read off the general solution:

$$\begin{aligned} v &= .5\,r \\ r &= r \quad \text{(free)} \end{aligned} \tag{4}$$

There are infinitely many fixed points, one for every value of r. We are at a fixed point, if the Vancouver population is half the size of the Richmond population. In this case we say that the system is in *equilibrium*.

To find the actual equilibrium point, i.e., the population vector at which the populations stabilize, we use our initial condition. Since we know that the total population always stays at 300+100=400, we get the additional equation $v + r = 400$. We get a new system of equations:

$$\begin{array}{rcrcr} v & - & .5r & = & 0 \\ v & + & r & = & 400 \end{array}$$

This system has the unique solution

$$\begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 133 \\ 267 \end{pmatrix}$$

(we have rounded these numbers). A close examination of Figure 3 reveals that the population vectors do seem to approach the vector $\vec{p}_\infty = \left(\begin{smallmatrix} 133 \\ 267 \end{smallmatrix}\right)$.

The line of fixed points

Let us see what happens if we start with different population vectors (but keeping the same dynamical system from Figure 1). In Figure 4 the trajectories of the population vectors are plotted for the initial conditions $\vec{p}_0 = \left(\begin{smallmatrix} 300 \\ 100 \end{smallmatrix}\right), \left(\begin{smallmatrix} 200 \\ 100 \end{smallmatrix}\right), \left(\begin{smallmatrix} 100 \\ 100 \end{smallmatrix}\right), \left(\begin{smallmatrix} 50 \\ 200 \end{smallmatrix}\right), \left(\begin{smallmatrix} 50 \\ 300 \end{smallmatrix}\right), \left(\begin{smallmatrix} 50 \\ 400 \end{smallmatrix}\right)$.

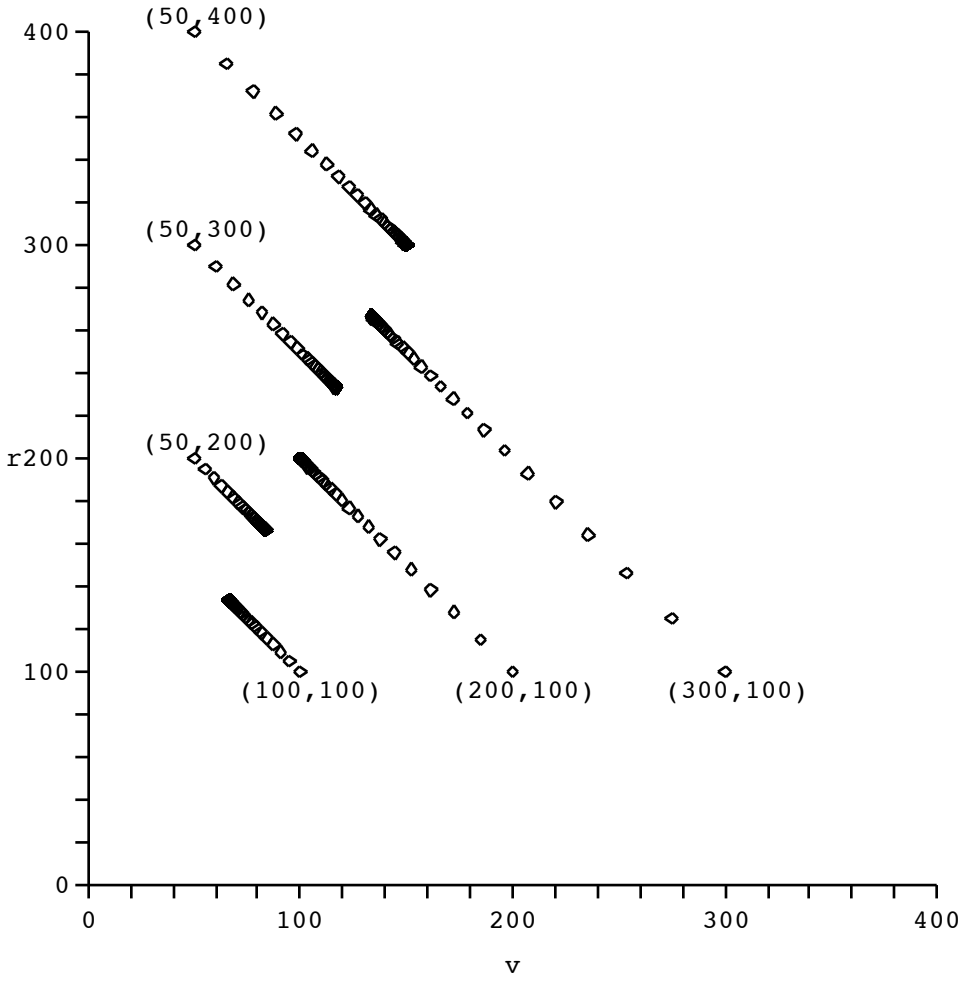
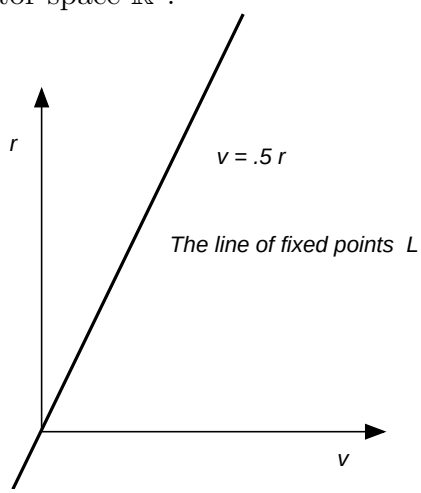


Figure 4: The phase portrait for various initial population vectors

We observe that in every case the population vector approaches a fixed point. The limiting point always satisfies the fixed point equation (2). The solution to (2) is always the same, as long as we are talking about the same transition matrix. The solution is (4). It describes a line L in the population vector space $\mathbb{R}^2$:



This line is the *line of fixed points* of the transition matrix A. In fact, all trajectories approach this line of fixed points as the years go by, as we see in Figure 4.

Let us do the above calculation starting with Equation (3) once again.

$$\begin{aligned} A\vec{p} &= \vec{p} \\ A\vec{p} &= I\vec{p} \\ I\vec{p} - A\vec{p} &= \vec{0} \\ (I - A)\vec{p} &= \vec{0} \end{aligned}$$

So the fixed points of A are the vectors in the null space of $I - A$.

1.2 Fibonacci's Example

Description of the dynamical system

Fibonacci considered the following problem. We breed rabbits, starting with one pair of rabbits. Every month, each pair of rabbits produces one pair of offspring. After one month, the offspring is adult, and will start reproduction. Describe the long term behaviour of the rabbit population. Again, we neglect

all kinds of effects. For example, no rabbit ever dies. And we always consider pairs of rabbits. Figure 5 summarizes the dynamics.

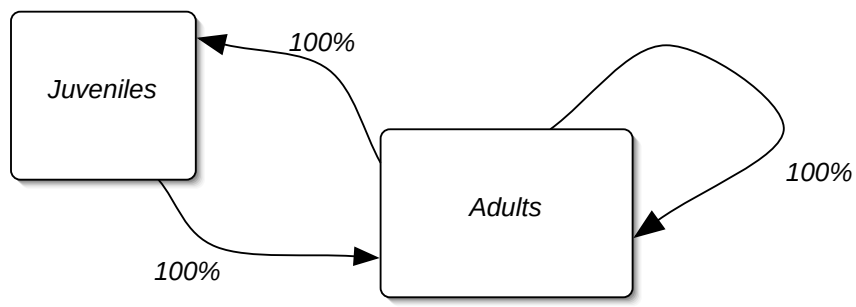


Figure 5: Rabbit breeding according to Fibonacci

Every month, all juveniles turn into adults, all adults give rise to an equal number (100%) of juveniles, and 100% of the adults stay around. This time, of course, the total number of rabbits is not constant, but increases quite rapidly.

Model using rabbit vectors

To model this dynamical system, we use the space of rabbit vectors $\vec{r} = \binom{j}{a} \in \mathbb{R}^2$. The first component of the rabbit vector gives the number of juvenile pairs, the second component the number of adult pairs of rabbits. Translating Figure 5 into equations, we get:

$$\begin{aligned} j_{n+1} &= \quad\quad a_n \\ a_{n+1} &= j_n + a_n \end{aligned}$$

In vector form

$$\begin{pmatrix} j_{n+1} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j_n \\ a_n \end{pmatrix}$$

or, even shorter

$$\vec{r}_{n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \vec{r}_n$$

This time, the transition matrix of the dynamical system is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

The initial condition is $\vec{r}_0 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, one pair of juvenile rabbits, no adult rabbits. We can summarize the dynamical system in the equations

$$\boxed{\begin{aligned} \vec{r}_{n+1} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \vec{r}_n \\ \vec{r}_0 &= \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \end{aligned}} \tag{5}$$

We can formally write down the solution to (5) as

$$\vec{r}_n = A^n \vec{r}_0$$

or explicitly:

$$\begin{pmatrix} j_n \\ a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

But this is not very illuminating: we do not know much about the powers of $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$, and we do not have a formula for the entries of $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)^n$. At this stage, the only thing we could do would be to keep multiplying $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ over and over with itself.

Starting the analysis

To get an idea of what is going on, let us calculate the first few rabbit vectors:

n	0	1	2	3	4	5	6	7	8
j_n	1	0	1	1	2	3	5	8	13
a_n	0	1	1	2	3	5	8	13	21

(6)

For example, the seventh rabbit vector is $\vec{r}_7 = \left(\begin{smallmatrix}8\\13\end{smallmatrix}\right)$. After 7 months there are 8 juvenile and 13 adult pairs of rabbits.

The sequence $(a_n)_{n=0,1,2\ldots} = (0, 1, 1, 2, 3, 5, 8 \ldots)$ is the famous Fibonacci sequence.

Let us plot these rabbit vectors, see Figure 6.

The method of eigenvalues

Rabbit vectors do not stabilize. Still, there is a similarity with the behaviour of the population vectors from Figure 3 or 4. Examining Figure 6, the rabbit vectors do seem to get closer and closer to a certain line, see Figure 7.

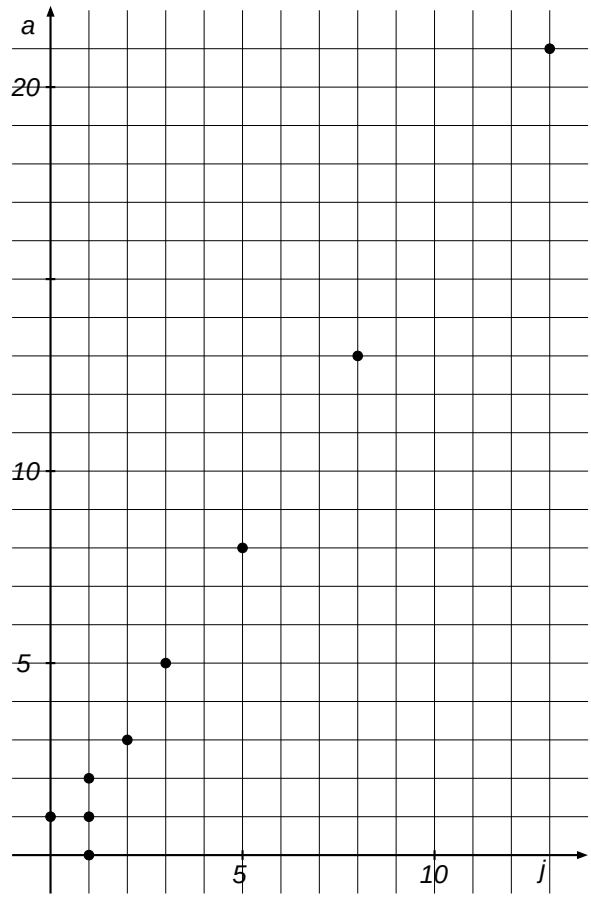


Figure 6: Rabbit vectors $\vec{r}_0, \ldots, \vec{r}_8$

The line L is called an *eigenspace* of our dynamical system, or better, an eigenspace of the transition matrix A (it is a one-dimensional vector space). Can we find the line L? The idea is similar to the fixed point equation (2), above.

Suppose that we have a rabbit vector $\vec{r}_n$, which is actually on the line L. With our initial condition, this will never happen, the rabbit vectors will never reach L, they only get closer and closer. But, suppose, anyway, that $\vec{r}_n$ is on L, then the next rabbit vector $\vec{r}_{n+1}$ (and all future rabbit vectors) will also be on L. That is precisely the distinguishing property of the line L. The actual rabbit vectors get closer and closer to L, but our 'imaginary'

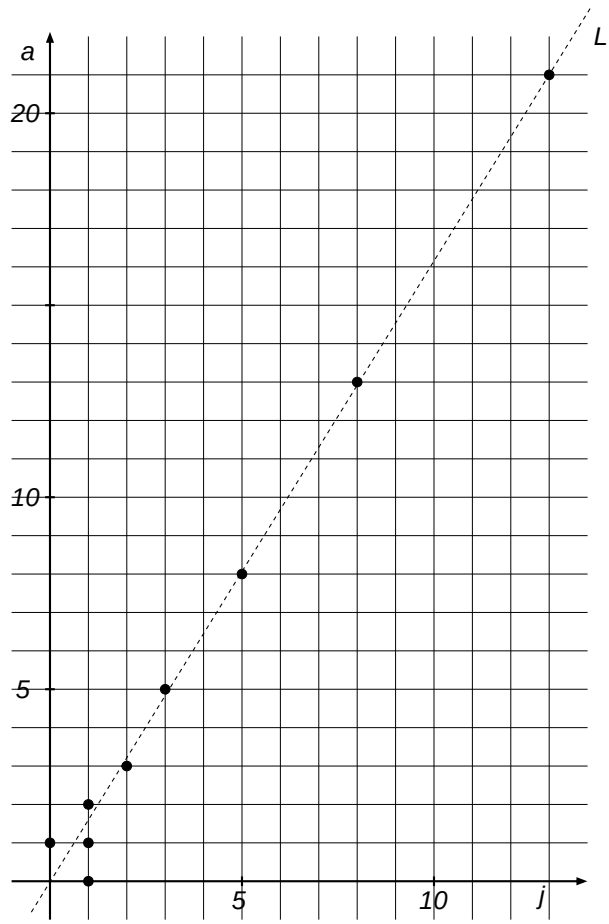


Figure 7: The line L

rabbit vector, once it is on L, stays on L:

$$\text{if } \vec{r}_n \text{ is on } L\text{, then } \vec{r}_{n+1} \text{ is also on } L$$

Or, using Equation (5) we can say

$$\text{if } \vec{r}_n \text{ is on } L\text{, then } A\vec{r}_n \text{ is also on } L$$

Let us drop the subscript, as we are anyway talking about an 'imaginary' rabbit vector, which is not on the list $\vec{r}_0, \vec{r}_1, \vec{r}_2, \ldots$.

$$\text{if } \vec{r} \text{ is on } L\text{, then } A\vec{r} \text{ is also on } L$$

Another way to say the same thing is

$$\text{both } \vec{r} \text{ and } A\vec{r} \text{ are on the same line}$$

Yet another way to say this is that

$$A\vec{r} \text{ is a scalar multiple of } \vec{r}$$

If $\vec{r} \neq \vec{0}$, then $\vec{r}$ will span the line L, and so if $A\vec{r}$ is also on L, then $A\vec{r}$ has to be a scalar multiple of $\vec{r}$. So there exists a scalar $\lambda \in \mathbb{R}$, such that

$$\boxed{\boxed{A\vec{r} = \lambda\vec{r}}} \tag{7}$$

Figure 8 gives a graphical representation of this equation.

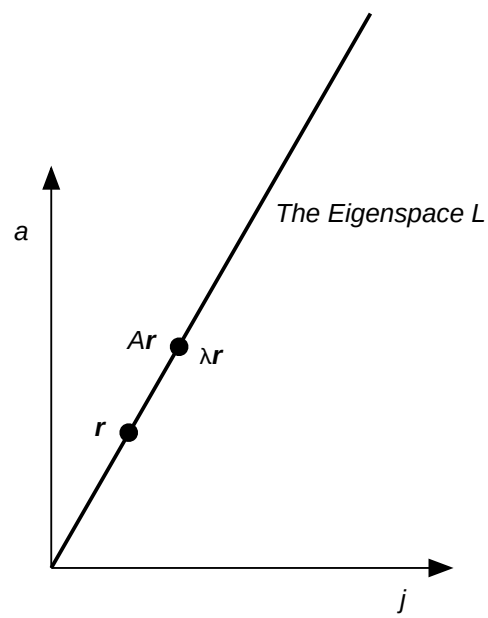


Figure 8: On the line L we have $A\vec{r} = \lambda\vec{r}$

With $\lambda = 1$, we get the fixed point equation (2). But there is one big difference: In Equation (2), we knew the value of λ, namely 1. This time we do not know the value of λ. In Equation (7), both $\vec{r}$ and λ are unknown. Only the matrix A is known, it is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Finding the eigenvalues

To solve (7), rewrite it as follows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} = \lambda \begin{pmatrix} j \\ a \end{pmatrix} \tag{8}$$

Now we do the same calculation as in (3):

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} &= \lambda \begin{pmatrix} j \\ a \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} \\
\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (9)
\end{aligned}$$

We first find λ, then the rabbit vector $\left(\begin{smallmatrix} j \\ a \end{smallmatrix}\right)$. Remember that we want the rabbit vector $\left(\begin{smallmatrix} j \\ a \end{smallmatrix}\right)$ to span the line L. So it has to be non-zero. Therefore, the matrix

$$\begin{pmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{pmatrix}$$

has to have some non-zero vector in its null space. This is the main idea. We need a non-zero vector to span the line L, so Equation (9) has to have a non-trivial solution vector $\left(\begin{smallmatrix} j \\ a \end{smallmatrix}\right)$. The coefficient matrix of the system (9), which is

$$\begin{pmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{pmatrix}$$

has to be singular. To check if this matrix is singular, we use the determinant. The matrix is singular, if the determinant is zero. So we are looking for values of λ, such that

$$\det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{pmatrix} = 0$$

The determinant is easily calculated:

$$\det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{pmatrix} = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1$$

The quadratic polynomial $\lambda^2 - \lambda - 1$ is called the **characteristic polynomial** of the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Its roots are called the **eigenvalues** of A.

Using the quadratic formula, we see that the solutions to

$$\lambda^2 - \lambda - 1 = 0 \tag{10}$$

are

$$\lambda_1 = \tfrac{1+\sqrt{5}}{2} \qquad \lambda_2 = \tfrac{1-\sqrt{5}}{2}$$

These are the two eigenvalues of the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Which one of these two eigenvalues are we looking for? Let us have another look at Figure 8. We see that $\lambda\vec{r}$ is larger than $\vec{r}$, so the scalar λ has to be greater than 1. So $\lambda_2 = \frac{1-\sqrt{5}}{2}$, which is less than zero, is wrong. So we have found the value of λ, namely

$$\lambda = \lambda_1 = \tfrac{1+\sqrt{5}}{2} \approx 1.618\,.$$

If a rabbit vector happens to be on the line L, then one month later it is still on the line L, but bigger by a factor of $\lambda_1 = \frac{1+\sqrt{5}}{2}$. For our actual rabbit vectors $\vec{r}_0, \vec{r}_1, \vec{r}_2 \ldots$ from our list (6), this is only approximately true. If n is large, then, $\vec{r}_{n+1} \approx \frac{1+\sqrt{5}}{2}\vec{r}_n$. As n gets larger, this approximation gets better.

Finding the eigenvectors

Let us now find the line L. Remember that a vector $\left(\begin{smallmatrix} j \\ a \end{smallmatrix}\right)$ will span the line L if it is a non-zero solution of Equation (8), or equivalently, (9). Plugging in the value $\lambda_1 = \frac{1+\sqrt{5}}{2}$, for λ, we obtain the equation

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} - 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

Let us row reduce the coefficient matrix

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} - 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix}$$

Note that we already knew that there was going to be a free variable: the whole point of choosing a λ solving (10), was to make sure that (9) has non-trivial solutions. The general solution to (11) is

$$\begin{pmatrix} j \\ a \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}-1}{2} a \\ a \end{pmatrix}$$

where a is the free variable. Any non-zero value for a gives us a spanning vector for L. To avoid fractions, let us pick $a = 2$. Thus, the vector

$$\vec{v}_1 = \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix}$$

spans the line L.

Because λ_1 and $\vec{v}_1$ solve the eigenvalue equation, we have

$$\boxed{\boxed{A\vec{v}_1 = \lambda_1 v_1}} \tag{12}$$

It may be worth checking that this is true:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} = \tfrac{1+\sqrt{5}}{2} \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} \tag{13}$$

In fact, both sides evaluate to $\left(\begin{smallmatrix} 2 \\ \sqrt{5}+1 \end{smallmatrix}\right)$.

The all important equation (12), or equivalently (13), is expressed in words by saying that

> $\vec{v}_1 = \left(\begin{smallmatrix} \sqrt{5}-1 \\ 2 \end{smallmatrix}\right)$ is an **eigenvector** of the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with corresponding eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$.

The slope of L in the ja-coordinate system is

$$\frac{a}{j} = \frac{2}{\sqrt{5}-1} = \frac{1+\sqrt{5}}{2} \approx 1.618$$

This agrees with Figure 7.

The line L consists of all solutions of (11).

> The line L is called the **eigenspace** of the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with respect to the eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$.

Qualitative description of the long-term behaviour

Because our rabbit vectors $\vec{r}_0, \vec{r}_1, \vec{r}_2, \ldots$ get closer and closer to the line L, the slope of these rabbit vectors approaches the value $\frac{1+\sqrt{5}}{2}$. We conclude that

$$\lim_{n\to\infty} \frac{a_n}{j_n} = \tfrac{1+\sqrt{5}}{2}\,.$$

Note. It is a coincidence that the slope of the eigenspace has the same value as the corresponding eigenvector.

We have succeeded in describing two major facts about the long term behaviour of the rabbit population:

(i) the slope of the rabbit vectors gets closer and closer to the slope of the eigenspace L, which means that the proportion of adult to juvenile rabbits approaches the value $\frac{1+\sqrt{5}}{2} \approx 1.618$, in the long run.

(ii) as n grows, the approximation $\vec{r}_{n+1} \approx \frac{1+\sqrt{5}}{2}\vec{r}_n$ gets better and better. Thus both the adult and the juvenile population grow approximately by a factor of $\frac{1+\sqrt{5}}{2} \approx 1.6.18$ every month.

The answer to the second question is given by the eigenvalue $\lambda_1 = \frac{1+\sqrt{5}}{2}$, the answer to the first question is given by the corresponding eigenspace $L = \operatorname{span} \vec{v}_1$, with slope $\frac{1+\sqrt{5}}{2}$.

The second eigenvalue

in fact, we can do much better. We can find exact formulas for the numbers of adult and juvenile rabbits in month n. Do do this, we need to study the other eigenvalue λ_2, which we discarded in the above discussion. Let us find the corresponding line of eigenvectors. Plugging in the value $\lambda_2 = \frac{1-\sqrt{5}}{2}$ into Equation 9, we get

$$\begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{1-\sqrt{5}}{2} - 1 \end{pmatrix} \begin{pmatrix} j \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$

This time the coefficient matrix reduces to

$$\begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{1-\sqrt{5}}{2} - 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix}$$

The general solution is

$$\begin{pmatrix} j \\ a \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{5}}{2}a \\ a \end{pmatrix}$$

This time, let us choose $a = -2$, to minimize fractions and negative numbers. We obtain the vector

$$\vec{v}_2 = \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix}$$

which spans a line M, through the origin, the eigenspace of the eigenvalue $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Thus we have found an eigenvector $\vec{v}_2$ of the matrix A with corresponding eigenvalue λ_2:

$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$

Again, it is a good idea to check our calculation:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix} = \frac{1-\sqrt{5}}{2} \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix}$$

This time, both sides evaluate to $\left(\begin{smallmatrix} -2 \\ \sqrt{5}-1 \end{smallmatrix}\right)$.

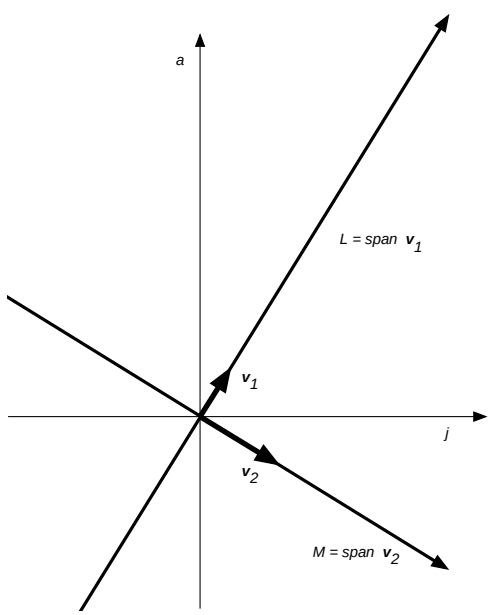


Figure 9: The two eigenspaces L and M

Let us recap: we set out to solve Equation (8). We found two solutions for λ: the eigenvalues λ_1 and λ_2, and for each of the two values of λ, we

found a whole line of solutions for $\binom{j}{a}$: the corresponding eigenspaces. In each eigenspace we picked a basis vector: $\vec{v}_1$ in L, the eigenspace of λ_1, and $\vec{v}_2$ in M, the eigenspace of λ_2. We have succeeded in finding a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$ of the space of rabbit vectors $\mathbb{R}^2$, consisting of eigenvectors for the transition matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Let us emphasize that the two eigenvalues are unique, but the corresponding eigenvectors are not. We are free to choose any spanning vectors $\vec{v}_1$, $\vec{v}_2$ of the eigenspaces we like, but there is nothing we can do about the values λ_1, λ_2 of the eigenvalues.

Exact solution

The point is that if we start with either the rabbit vector $\vec{v}_1$ or $\vec{v}_2$, the future behaviour of the dynamical system is easy to describe: every time we multiply $\vec{v}_1$ by the transition matrix A, we are really just multiplying by the scalar λ_1, so

$$A^n \vec{v}_1 = \lambda_1^n \vec{v}_1 \,, \tag{15}$$

by applying (12) n times over. Writing this out, it becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} = (\tfrac{1+\sqrt{5}}{2})^n \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix}$$

The rabbit vector retains its direction, but gets longer by a factor of λ_1, at each iteration.

By the same token, if we start with $\vec{v}_2$, the future is clear:

$$A^n \vec{v}_2 = \lambda_2^n \vec{v}_2 \,, \tag{16}$$

or written out:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix} = (\tfrac{1-\sqrt{5}}{2})^n \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix}$$

Because $\lambda_2 = \frac{1-\sqrt{5}}{2}$ is negative, the rabbit vector changes direction, each time we multiply by λ_2. Also, because the absolute value of $\lambda_2 = \frac{1-\sqrt{5}}{2}$ is less than 1, the vector gets shorter at each iteration. But it always stays on the line M.

Now, the crucial fact is that every rabbit vector $\vec{r}$ is a linear combination of the two basic rabbit vectors $\vec{v}_1$ and $\vec{v}_2$. Since we know what happens to

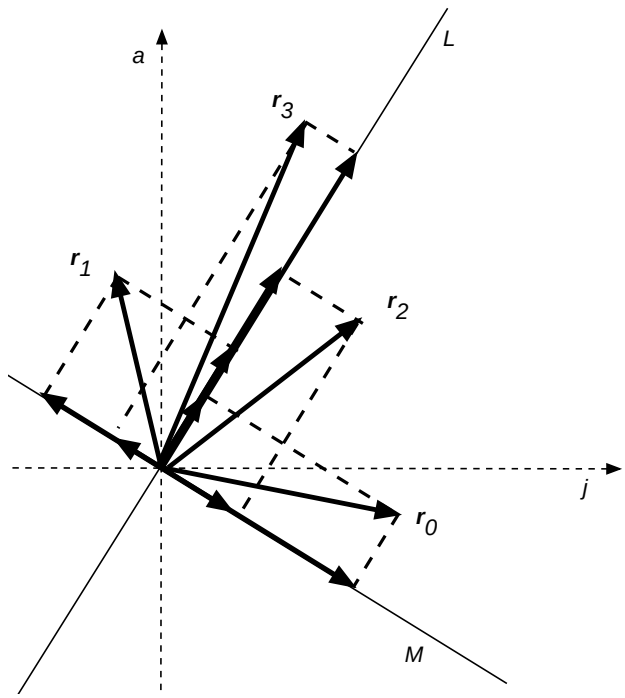


Figure 10: Three iterations of a rabbit vector $\vec{r}_0$. Notice how the M-component gets smaller and changes direction each time, and how the L-component keeps growing

$\vec{v}_1$ and $\vec{v}_2$, and the transition from on month to the next is *linear*, we know what happens to every rabbit vector. if, say

$$\vec{r}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 \tag{17}$$

Then

$$\begin{aligned}
\vec{r}_n &= A^n\vec{r}_0 \\
&= A^n(c_1\vec{v}_1 + c_2\vec{v}_2) && \text{plug in (17)} \\
&= c_1A^n\vec{v}_1 + c_2A^n\vec{v}_2 && \text{rules for matrix multiplication} \\
&= c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2 && \text{from (15) and (16)} \\
&= \lambda_1^n c_1\vec{v}_1 + \lambda_2^n c_2\vec{v}_2
\end{aligned}$$

Thus we have solved our dynamical system:

$$\begin{pmatrix} j_n \\ a_n \end{pmatrix} = (\tfrac{1+\sqrt{5}}{2})^n c_1 \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} + (\tfrac{1-\sqrt{5}}{2})^n c_2 \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix} \tag{18}$$

In Figure 10, we have tried to sketch the situation: we have started with a vector $\vec{r}_0$, and have plotted three iterations of the dynamical system. Each of the four rabbit vectors, we have written as a sum of its component along L and its component along M. For example, $\vec{r}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$, so $c_1\vec{v}_1$ is the component along L and $c_2\vec{v}_2$, the component along M. The vector $c_1\vec{v}_1$ stays on L, the vector $c_2\vec{v}_2$ stays on M, but the linear combination $\vec{r}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$ moves around in the plane.

Finding the constants

The only thing still missing from (18) are the values of the constants c_1 and c_2. These we get from the initial condition, Equation (17), or by plugging in $n = 0$ into (18):

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix}$$

This inhomogeneous linear system is solved using the augmented coefficient matrix:

$$\left(\begin{array}{cc|c} \sqrt{5}-1 & 1+\sqrt{5} & 1 \\ 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ \sqrt{5}-1 & 1+\sqrt{5} & 1 \end{array}\right) \to$$

$$\to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ \sqrt{5} & \sqrt{5} & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 5 & 5 & \sqrt{5} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & \frac{\sqrt{5}}{10} \\ 0 & 1 & \frac{\sqrt{5}}{10} \end{array}\right)$$

The unique solution is

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}}{10} \\ \frac{\sqrt{5}}{10} \end{pmatrix}$$

Plug this into (18) to obtain the final formula for $\vec{r}_n$

$$\boxed{\begin{pmatrix} j_n \\ a_n \end{pmatrix} = \tfrac{\sqrt{5}}{10}\left((\tfrac{1+\sqrt{5}}{2})^n \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} + (\tfrac{1-\sqrt{5}}{2})^n \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix} \right)} \tag{19}$$

For example, just looking at the second components, we get a formula for the Fibonacci numbers:

$$a_n = \tfrac{\sqrt{5}}{5}\left((\tfrac{1+\sqrt{5}}{2})^n - (\tfrac{1-\sqrt{5}}{2})^n \right)$$

More detailed analysis

Now that we have Formula (19), we can give a more rigourous analysis of the long term behaviour of the dynamical system. The most important information is contained in the two eigenvalues.

Since $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$ is greater than 1, we have

$$\lim_{n\to\infty} \lambda_1^n = \infty\,.$$

Since $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$ has absolute value less than 1, we have

$$\lim_{n\to\infty} \lambda_2^n = 0\,.$$

Hence, the second summand of (19) converges to the zero vector:

$$\lim_{n\to\infty} \left((\tfrac{1-\sqrt{5}}{2})^n \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so contributes less and less to $\vec{r}_n$, as n gets larger. So we can write

$$\begin{pmatrix} j_n \\ a_n \end{pmatrix} \approx \tfrac{\sqrt{5}}{10} (\tfrac{1+\sqrt{5}}{2})^n \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix}$$

Thus we confirm what we saw above: in the long run, rabbit vectors just get rescaled by the factor $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$, and

$$\lim_{n\to\infty} \tfrac{a_n}{j_n} = \tfrac{2}{\sqrt{5}-1} = \tfrac{1+\sqrt{5}}{2} \approx 1.618\,.$$

We also get an approximate formula for Fibonacci numbers:

$$a_n \approx \tfrac{\sqrt{5}}{2} (\tfrac{1+\sqrt{5}}{2})^n$$

and the limiting behaviour of successive Fibonacci numbers:

$$\lim_{n\to\infty} \tfrac{a_{n+1}}{a_n} = \tfrac{1+\sqrt{5}}{2}$$

All of this is because the bigger eigenvalue λ_1 ends up dominating (19).

The Golden Ratio

The number $\frac{1+\sqrt{5}}{2}$ is known as the *golden ratio*. It is one of the most famous numbers in the history of art and mathematics (see `wikipedia` for more information).

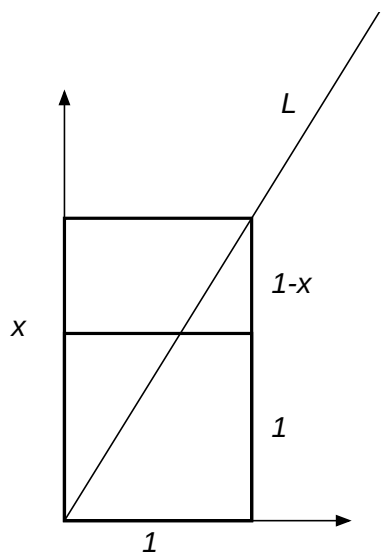


Figure 11: A golden rectangle. The small rectangle obtained by cutting off a square from the golden rectangle is another golden rectangle. The diagonal of the golden rectangle is the eigenspace L

A *golden rectangle*, is a rectangle with the property that if you cut off a square, the remaining rectangle is similar to, i.e., has the same shape as, the original rectangle. Referring to Figure 11, suppose the shorter side of the golden rectangle has length 1 and the larger length x, then

$$x : 1 = 1 : 1 - x$$

the longer side the of the big rectangle x, is in the same relation to the shorter side 1, as the longer side 1, of the small rectangle is to its shorter side, $1 - x$. Rewriting this equality of relations in terms of fractions gives

$$x = \frac{1}{1 - x}$$

or

$$x^2 - x - 1 = 0$$

which is the characteristic equation (10) of the transition matrix A of the Fibonacci dynamical system. The positive solution is

$$x = \frac{1 + \sqrt{5}}{2}$$

which is the ratio of the long side to the short side of the golden rectangle, and is therefore called the golden ratio.

1.3 Predator-Prey System

Frogs and flies

Predator-Prey systems are often used as examples of linear dynamical systems, even though this is usually not realistic. Here, we will try to describe an example where a linear dynamical system may not be too far fetched (the numbers are made up).

We follow the polulation growth of two animal species, frogs and flies in a certain habitat. The frog population measures in millions and the fly population in billions, so we will take millions (10^6) as units for the size of the frog population and billions (10^9) as units to measure the size of the fly population. We check the population size once a year.

Flies have, in the absence of frogs, a growth rate of 122%. This means that the fly population will be larger by 22% after one year, if it not diminished by the presence of frogs.

Frogs will prey on flies, and we assume that the decrease of the fly population as a result of this, is directly proportional to the number of frogs present. In fact, the impact of the predation by frogs on the fly population is -36%. This means that the presence of 100 million frogs will diminish the fly population by 36 billion. In other words, every frog will eat 360 flies per year. The frogs are perfect predators: they will hunt down and eat the required number of flies, no matter how scarce they are. They will not eat more flies if flies are abundant.

Thus, there is no direct effect of a change of the fly population on the number of frogs. The way the fly population affects the frog population is by affecting the reproduction rate of frogs. To justify that the increase of the frog population due to the presence of flies depends linearly on the number of

flies (in other words, that the increase of the frog population is proportional to the number of flies), we assume that the frogs pretty much saturate the ponds and lakes in the habitat with frog eggs and tadpoles each spring, and that the tadpoles (which also eat flies) are not good hunters at all. The number of tadpoles hatching in the spring is *constant*, and the number of tadpoles surviving to grow into frogs depends directly on the number of flies that are around. If there are three times as many flies, there will be three times as many surviving tadpoles. We will take the increase of the frog population in response to the fly population to be 24%. Thus, 100 billion flies will result in 24 million tadpoles surviving to become frogs.

[Note that this increase is assumed to be independent of the number of frogs. This may be the most unrealistic aspect of this model. More realistically, the number of tadpoles present in the spring would depend on the number of frogs present, and the survival rate of tadpoles would depend on the fly population. So the increase of the frog population as a result of the presence of flies would depend on both the frog and fly population, but this effect is not linear. The simplest dependence would be proportional to the product of the two populations.]

Finally, we need the growth rate of the frog population in the absence of flies. It is 38%. This means that if there are no flies for the tadpoles or frogs to feed on, the frog population will decrease by 62% every year.

The problem we will solve is to find out what the long term behaviour of the frog and fly populations are. Under what conditions do these two species thrive, or under what conditions do they become extinct.

The model

We model the populations using frog-fly-vectors, or f-vectors. The f-vectors are vectors $\binom{g}{y} \in \mathbb{R}^2$, where g counts the units of frogs (so $g = 3$ means 3 million frogs), and y counts the units of flies (so $y = 5$ means 5 billion flies). The values of g and y next year, i.e. g_{n+1} and y_{n+1} are given in terms of the values of g and s this year, by

$$\begin{aligned} g_{n+1} &= .38g_n + .24y_n \\ y_{n+1} &= -.36g_n + 1.22y_n \end{aligned}$$

Or in matrix-vector notation:

$$\vec{f}_{n+1} = \begin{pmatrix} .38 & .24 \\ -.36 & 1.22 \end{pmatrix} \vec{f}_n$$

The transition matrix of this dynamical system is

$$A = \begin{pmatrix} .38 & .24 \\ -.36 & 1.22 \end{pmatrix}$$

and the frog-fly vector after n years is given in terms of the initial frog-fly vector by

$$\vec{f}_n = A^n \vec{f}_0$$

Solving the system

To find the eigenvalues of the transition matrix A, find its characteristic polynomial:

$$\begin{aligned} \det \begin{pmatrix} \lambda - .38 & -.24 \\ .36 & \lambda - 1.22 \end{pmatrix} &= (\lambda - .38)(\lambda - 1.22) + .36 \cdot .24 \\ &= \lambda^2 - (.38 + 1.22)\lambda + .38 \cdot 1.22 + .36 \cdot .24 \\ &= \lambda^2 - 1.6\lambda + .55 \end{aligned}$$

Use the quadratic formula to solve this quadratic equation for λ.

$$\begin{aligned} \lambda &= \tfrac{1}{2}(1.6 \pm \sqrt{1.6^2 - 4 \cdot .55}) \\ &= \tfrac{1}{2}(1.6 \pm \sqrt{2.56 - 2.2}) \\ &= \tfrac{1}{2}(1.6 \pm \sqrt{.36}) \\ &= \tfrac{1}{2}(1.6 \pm .6) \\ &= .8 \pm .3 \end{aligned}$$

This gives us the two eigenvalues

$$\lambda_1 = 1.1 \qquad \lambda_2 = .5$$

The eigenspace E_1 of λ_1 is the null space of the matrix

$$\begin{pmatrix} 1.1 - .38 & -.24 \\ .36 & 1.1 - 1.22 \end{pmatrix} = \begin{pmatrix} .72 & -.24 \\ .36 & -.12 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}$$

One free variable means that E_1 is one-dimensional. A convenient basis for E_1 is the vector

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

so

$$E_1 = \operatorname{span} \vec{v}_1 = \operatorname{span} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The eigenspace E_2 of λ_2 is the null space of the matrix

$$\begin{pmatrix} .5 - .38 & -.24 \\ .36 & .5 - 1.22 \end{pmatrix} = \begin{pmatrix} .12 & -.24 \\ .36 & -.72 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

One free variable means that E_2 is one-dimensional. A convenient basis for E_2 is the vector

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

so

$$E_2 = \operatorname{span} \vec{v}_2 = \operatorname{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We note that $(\vec{v}_1, \vec{v}_2)$ is a basis for the space $\mathbb{R}^2$ of all f-vectors. Now we are ready to write down the general solution. If the initial f vector is

$$\vec{f}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 \tag{20}$$

then the n-th f-vector is

$$\begin{aligned} \vec{f}_n &= A^n \vec{f}_0 \\ &= A^n(c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1 A^n \vec{v}_1 + c_2 A^n \vec{v}_2 \\ &= c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_2^n \vec{v}_2 \\ &= 1.1^n c_1 \left(\tfrac{1}{3}\right) + 0.5^n c_2 \left(\tfrac{2}{1}\right) \\ &= \begin{pmatrix} (1.1)^n c_1 + 2(.5)^n c_2 \\ 3(1.1)^n c_1 + (.5)^n c_2 \end{pmatrix} \end{aligned}$$

Thus, the frog population is given by

$$g_n = (1.1)^n c_1 + 2(.5)^n c_2$$

and the fly population by

$$y_n = 3(1.1)^n c_1 + (.5)^n c_2$$

The constants c_1 and c_2 are to be determined from $\vec{f}_0$, by solving (20). Since no initial condition is given, we cannot solve for c_1 and c_2, so we leave the solution in the *general* form:

$$\begin{pmatrix} g_n \\ v_n \end{pmatrix} = 1.1^n c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0.5^n c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Discussion of the solution

Of course, the actual frog-fly vector $\vec{f_0}$ has to have positive components. That does not mean that c_1 and c_2 have to be positive.

Let us remark that $\lim_{n\to\infty}(1.1)^n = \infty$ and $\lim_{n\to\infty}(.5)^n = 0$. So, if $c_1 > 0$, then, no matter what c_2 is, eventually the term $1.1^n c_1\left(\begin{smallmatrix}1\\3\end{smallmatrix}\right)$ will outgrow the term $0.5^n c_2\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$ in the formula

$$\vec{f_n} = 1.1^n c_1\left(\begin{smallmatrix}1\\3\end{smallmatrix}\right) + 0.5^n c_2\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$$

and both the frog and the fly population will grow at a rate of 1.1 or 110% per year. Moreover, the proportion of units of frogs to units of flies will get closer and closer to $1:3$.

On the other hand, if $c_1 < 0$, then

$$\lim_{n\to\infty} 1.1^n c_1 \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}-\infty\\-\infty\end{pmatrix}$$

which means that both populations will die out. The boundary case, when $c_1 = 0$ implies that $\vec{f_n} = 0.5^n c_2\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$, and the populations will also die out eventually, but much slower.

What is the meaning of $c_1 > 0$ in terms of the initial f-vector $\vec{f_0}$? For this, we go back to Equation (20), which we rewrite as

$$\begin{pmatrix}g_0\\y_0\end{pmatrix} = c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\begin{pmatrix}2\\1\end{pmatrix}$$

or as

$$\begin{aligned} g_0 &= c_1 + 2c_2 \\ y_0 &= 3c_1 + c_2 \end{aligned}$$

Let us eliminate c_2 from these two equations by solving the second equation for c_2, and plugging the result into the first equation. We get:

$$g_0 = c_1 + 2(y_0 - 3c_1)$$

which we rearrange to

$$2y_0 - g_0 = 5c_1$$

So now we see that $c_1 > 0$ if and only if $2y_0 - g_0 > 0$, or, in other words, $2y_0 > g_0$, or $y_0 > \frac{1}{2}g_0$.

Remember that $c_1 > 0$ was the case of survival of both species. We conclude that if initially there are more than half as many units of flies as units of frogs, both species will thrive and in the long run, they will both grow at a rate of 110% per year. This means that in the beginning there need to be more than 500 million flies for every 1 million frogs. Moreover, in the long run, there will be approximately 3 billion flies for every 1 billion frogs.

The phase portrait

We can get a better picture if we consider the phase portrait of the dynamical system. The phase portrait is a graphical representation of the long term behaviour of the dynamical system for different initial conditions. Thus Figure 4 is a phase portrait of the Vancouver-Richmond dynamical system.

The eigenspace analysis will help us sketch the phase portrait. We start by graphing the eigenspaces (Figure 12).

Some of the trajectories are very easy to sketch: if an f-vector is on one of the eigenspaces, it stays there. If it is on E_1, it gets bigger, as the years go by (by a factor of 1.1 each year), if it is on E_2, it gets smaller every year (by a factor of 0.5). On E_1, the f-vectors move away from the origin, on E_2, they move towards the origin. We indicate these observations in Figure 13.

What do the other trajectories look like? They are not lines, but curves. We can determine the asymptotic behaviour of these trajectories. Let us recall the solution of the dynamical system:

$$\vec{f_n} = 1.1^n c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0.5^n c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

As we have discussed, in the positive time line (as $n \to +\infty$), the first term $1.1^n c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ gets larger, whereas the second term $0.5^n c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ goes to zero (because $1.1 > 1$ and $0.5 < 1$). So the vector $\vec{f_n}$ gets closer and closer to the line spanned by $\left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right)$, which is E_1. In other words, E_1 is an asymptote of all the trajectories.

Let us see what happens as time runs backwards, i.e., as $n \to -\infty$. We have

$$\lim_{n\to-\infty} 1.1^n = \lim_{n\to\infty} 1.1^{-n} = \lim_{n\to\infty} (\frac{1}{1.1})^n = \lim_{n\to\infty} (\frac{10}{11})^n = 0$$

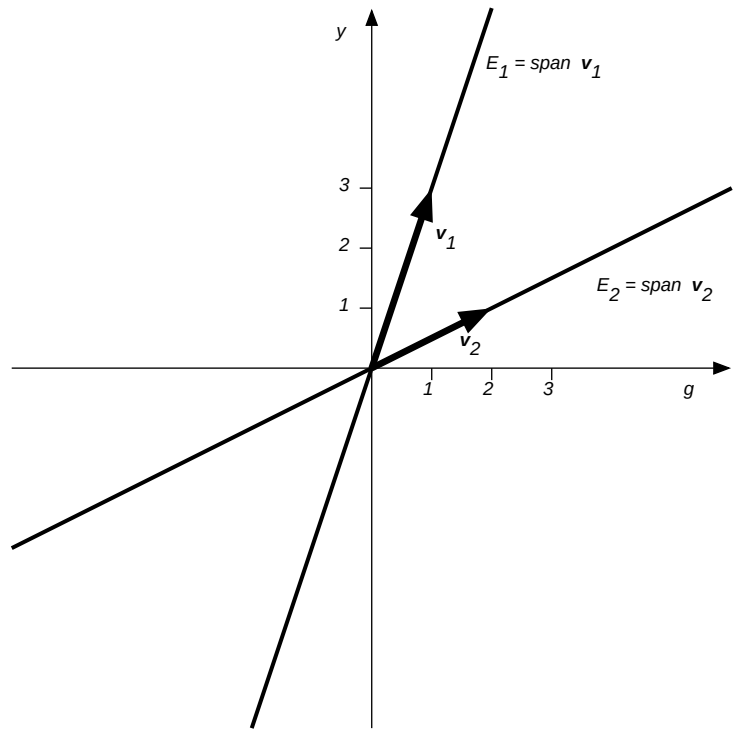


Figure 12: The two eigenspaces

and

$$\lim_{n\to-\infty} 0.5^n = \lim_{n\to\infty} 0.5^{-n} = \lim_{n\to\infty} (\frac{1}{0.5})^n = \lim_{n\to\infty} 2^n = \infty$$

So along the negative time line, the behaviour is exactly the opposite: the term $1.1^n c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ goes to zero and the term $0.5^n c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ gets bigger and bigger. Hence, as time goes backwards, the vector $\vec{f_n}$ gets closer and closer to the line spanned by $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, which is E_2. So E_2 is also an asymptote of all the trajectories. We can thus roughly sketch a few more trajectories as in Figure 14.

The physically meaningful range of f-vectors (the first quadrant, where both g and y are greater than zero) is shaded light grey, the region leading to eventual extinction is shaded dark grey. We see that, in fact, the flies become extinct before the frogs do. (But, of course, after the trajectory

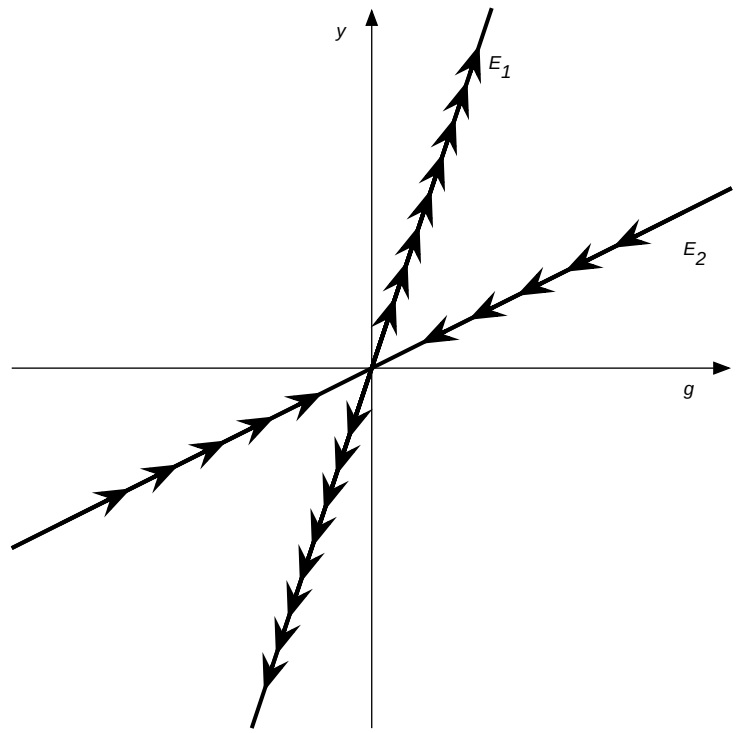


Figure 13: The trajectories along the two eigenspaces

passes through the g-axis at $y = 0$, it ceases to be meaningful, as y, the number of flies cannot become negative).

We also see that if $\vec{f_0}$ starts out very close to E_2, so that there are only slightly more flies than necessary to keep the populations from going extinct, the populations will first decrease, before they recover and start to grow.

Figure 15 is produced by a computer and contains more realistic trajectories. But note that we can get a pretty good qualitative picture without the help of the computer.

The method of diagonalization

We have used the standard coordinate system in the space of frog-fly vectors. The standard coordinates are called g and y. The eigenbasis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$ defines a new coordinate system in the vector space of frog-fly vectors. Let

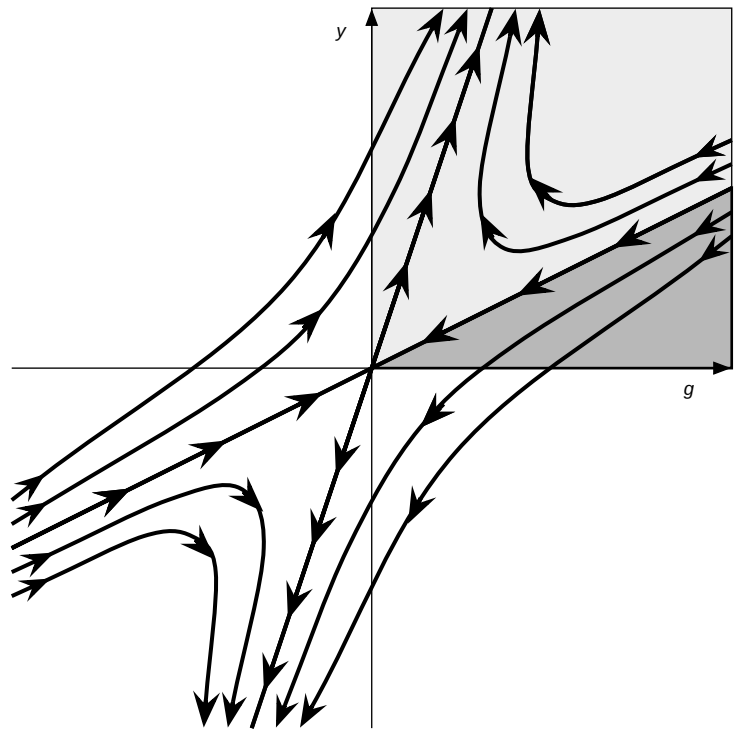


Figure 14: Very rough sketch of some more trajectories

us call T the linear transition transformation. The transformation T takes frog-fly vectors this year as input, and outputs the frog-fly vectors of next year.

Recall the formula

$$\boxed{\boxed{[T]_{\mathcal{E}} = P\ [T]_{\mathcal{B}}\ P^{-1}}} \tag{21}$$

Here $[T]_{\mathcal{E}}$ is the standard matrix of T. It is the matrix A, which was given by the description of the dynamical system:

$$[T]_{\mathcal{E}} = A = \begin{pmatrix} .38 & .24 \\ -.36 & 1.22 \end{pmatrix}$$

The matrix $[T]_{\mathcal{B}}$ has a very nice form: Note that

$$T(\vec{v}_1) = A\vec{v}_1 = \lambda_1\vec{v}_1 = 1.1\vec{v}_1 = 1.1\vec{v}_1 + 0\vec{v}_2$$

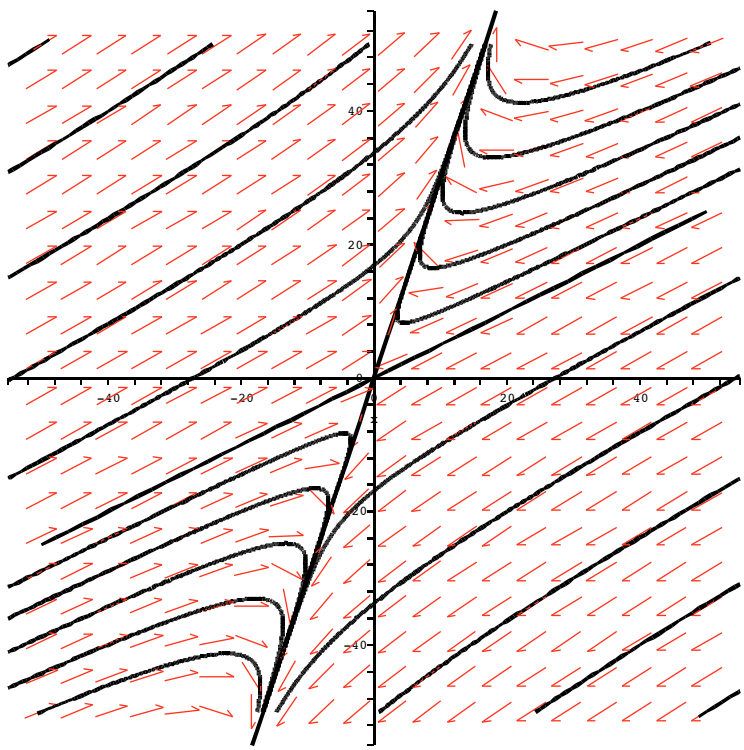


Figure 15: A more realistic phase portrait done by computer. The black curves are the trajectories, the little red arrows give the direction of the trajectories.

because $\vec{v}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 1.1$. Thus

$$[T(\vec{v}_1)]_{\mathcal{B}} = \begin{pmatrix} 1.1 \\ 0 \end{pmatrix} \tag{22}$$

For the same reason, we have

$$T(\vec{v}_2) = A\vec{v}_2 = \lambda_2\vec{v}_2 = 0.5\vec{v}_2 = 0\vec{v}_1 + 0.5\vec{v}_2$$

so

$$[T(\vec{v}_2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \tag{23}$$

Equations (22) and (23) give us the two columns of $[T]_{\mathcal{B}}$:

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1.1 & 0 \\ 0 & 0.5 \end{pmatrix}$$

The matrix of T with respect to the eigenbasis $\mathcal{B}$ is diagonal with the eigenvalues on the diagonal.

The matrix P is the matrix which turns $\mathcal{B}$-coordinate vectors into $\mathcal{E}$-coordinate vectors, it has the vectors $\vec{v}_1$ and $\vec{v}_2$ as columns:

$$P = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

Its inverse is

$$P^{-1} = \tfrac{1}{5} \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$$

Now Formula (21) reads

$$\begin{pmatrix} .38 & .24 \\ -.36 & 1.22 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1.1 & 0 \\ 0 & 0.5 \end{pmatrix} \tfrac{1}{5} \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix} \tag{24}$$

(It is a good idea to check that this is correct.) The whole point of calculating the eigenvalues and the eigenvectors is to obtain the factorization (24). We also write it like this:

$$\boxed{A = P\,D\,P^{-1}} \tag{25}$$

where D stands for diagonal.

Remember that to solve the dynamical system it is enough to calculate the powers of the matrix A:

$$\vec{f}_n = A^n \vec{f}_0 \tag{26}$$

The factorization (25) is useful. For example, for A^3:

$$\begin{aligned} A^3 &= (P\,D\,P^{-1})^3 \\ &= (P\,D\,P^{-1})(P\,D\,P^{-1})(P\,D\,P^{-1}) \end{aligned}$$

$$
\begin{aligned}
&= P\,D(P^{-1}\,P)D(P^{-1}\,P)DP^{-1} && \text{by re-bracketing}\\
&= P\,D\,I\,D\,I\,DP^{-1} && \text{because } P^{-1}P = I\\
&= P\,D\,D\,DP^{-1} && \text{the identity matrix } I \text{ has no effect}\\
&= P\,D^3\,P^{-1}
\end{aligned}
$$

The same works for every n:

$$\boxed{\boxed{A^n = P\,D^n\,P^{-1}}} \tag{27}$$

But the powers of a diagonal matrix are easy:

$$
\begin{aligned}
D^3 = \begin{pmatrix} 1.1 & 0 \\ 0 & 0.5 \end{pmatrix}^3 &= \begin{pmatrix} 1.1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} 1.1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} 1.1 & 0 \\ 0 & 0.5 \end{pmatrix}\\
&= \begin{pmatrix} 1.1^2 & 0 \\ 0 & 0.5^2 \end{pmatrix}\begin{pmatrix} 1.1 & 0 \\ 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 1.1^3 & 0 \\ 0 & 0.5^3 \end{pmatrix}
\end{aligned} \tag{28}
$$

and in general

$$D^n = \begin{pmatrix} 1.1^n & 0 \\ 0 & 0.5^n \end{pmatrix}$$

Plugging this into (27) gives

$$
\begin{aligned}
\begin{pmatrix} .38 & .24 \\ -.36 & 1.22 \end{pmatrix}^n &= A^n = P\,D^n\,P^{-1}\\
&= P\begin{pmatrix} 1.1^n & 0 \\ 0 & 0.5^n \end{pmatrix}P^{-1}\\
&= \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1.1^n & 0 \\ 0 & 0.5^n \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}\\
&= \begin{pmatrix} 1.1^n & 2\cdot 0.5^n \\ 3\cdot 1.1^n & 0.5^n \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}\\
&= \tfrac{1}{5}\begin{pmatrix} -1.1^n + 6\cdot 0.5^n & 2\cdot 1.1^n - 2\cdot 0.5^n \\ -3\cdot 1.1^n + 3\cdot 0.5^n & 6\cdot 1.1^n - 0.5^n \end{pmatrix}
\end{aligned}
$$

Thus, we have finally succeeded in computing the powers of the transition matrix A. We now simply plug this result into (26) to get the general formula for $\vec{f_n}$ in terms of $\vec{f_0}$

$$\vec{f_n} = \begin{pmatrix} g_n \\ y_n \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} -1.1^n + 6\cdot 0.5^n & 2\cdot 1.1^n - 2\cdot 0.5^n \\ -3\cdot 1.1^n + 3\cdot 0.5^n & 6\cdot 1.1^n - 0.5^n \end{pmatrix}\begin{pmatrix} g_0 \\ y_0 \end{pmatrix}$$

$$
\begin{aligned}
&= \tfrac{1}{5}\begin{pmatrix} -g_0 1.1^n + 6g_0 0.5^n + 2y_0 1.1^n - 2y_0 0.5^n \\ -3g_0 1.1^n + 3g_0 0.5^n + 6y_0 1.1^n - y_0 0.5^n \end{pmatrix} \\
&= \tfrac{1}{5}\begin{pmatrix} (-g_0 + 2y_0)1.1^n + (6g_0 - 2y_0)0.5^n \\ (-3g_0 + 6y_0)1.1^n + (3g_0 - y_0)0.5^n \end{pmatrix} \\
&= \tfrac{1}{5}1.1^n \begin{pmatrix} -g_0 + 2y_0 \\ -3g_0 + 6y_0 \end{pmatrix} + \tfrac{1}{5}0.5^n \begin{pmatrix} 6g_0 - 2y_0 \\ 3g_0 - y_0 \end{pmatrix}
\end{aligned}
$$

From this formula we can also read off the condition that the populations do not die out: the vector multiplying the expression 1.1^n has to have positive components, which means that $-g_0 + 2y_0 > 0$ and $-3g_0 + 6y_0 > 0$. These inequalities are both equivalent to $2y_0 > g_0$, which is the same answer we obtained by looking at the phase portrait.

To summarize, let us write down the frog and fly populations after n years in terms of the initial populations:

$$
\begin{aligned}
g_n &= \tfrac{1}{5}(-\ \ g_0 + 2y_0)1.1^n + \tfrac{1}{5}(6g_0 - 2y_0)0.5^n \\
y_n &= \tfrac{1}{5}(-3g_0 + 6y_0)1.1^n + \tfrac{1}{5}(3g_0 - \ \ y_0)0.5^n
\end{aligned}
$$

Concluding remarks

A more commonly used model for the predator-prey dynamics, which is often more realistic, is the so-called Lotka-Volterra Model. In our example, it would look something like this:

$$
\begin{aligned}
g_{n+1} &= \quad .38g_n + \ \ .24g_n y_n \\
y_{n+1} &= -.36g_n y_n + 1.22y_n
\end{aligned}
$$

The growth terms in the absence of the other species are the same as in our linear model, but the cross terms which describe how the two species influence each other are proportional to the product $g_n y_n$. This expresses the fact that for predation to occur, the predator actually has to encounter the prey, and the likelihood of that happening depends on the product of the two population sizes. But this model is not linear, the vector $\binom{g_{n+1}}{y_{n+1}}$ cannot be obtained from $\binom{g_n}{y_n}$ by multiplying by a matrix with real numbers as entries. The transition transformation is not a linear transformation, the dynamical system is not linear. It cannot be solved by our methods.

1.4 Summary of the Method

The discrete dynamical system is described by the formula

$$\vec{x}_{n+1} = A\vec{x}_n \tag{29}$$

Here $\vec{x} \in \mathbb{R}^k$ is the *state vector* and $A \in M_{k\times k}(\mathbb{R})$ is the *transition matrix*. The solution to (29) is

$$\vec{x}_n = A^n\vec{x}_0 \tag{30}$$

where $\vec{x}_0$ is the *initial state*.

To solve the system (or to make the solution (30) explicit), we use the method of eigenvalues. We start with the *characteristic polynomial* of the matrix A:

$$\det(\lambda I_k - A)$$

This is a polynomial of degree k in the variable λ with coefficients in $\mathbb{R}$. The *eigenvalues* of A are the solutions of the *characteristic equation*

$$\det(\lambda I_k - A) = 0$$

Suppose the eigenvalues are $\lambda_1, \ldots, \lambda_r$ (the number of eigenvalues r satisfies $r \leq k$). For each eigenvalue λ_j the null space of the matrix

$$\lambda_j I - A$$

or the solution space of the homogeneous system of equations

$$(\lambda_j I - A)\vec{x} = \vec{0}$$

is the *eigenspace* corresponding to the eigenvalue λ_j, notation E_j. We find a basis $\mathcal{B}_j$ of E_j.

Fact 1 If you put all the basis $\mathcal{B}_1 \ldots, \mathcal{B}_r$ of all the eigenspaces $E_1, \ldots, E_r$ of all the eigenvalues $\lambda_1, \ldots, \lambda_r$ together in one set $\mathcal{B}$, you get a linearly independent set of vectors in $\mathbb{R}^k$.

It may or may not be the case that the vectors in the resulting set $\mathcal{B}$ span $\mathbb{R}^k$. There may be fewer than k vectors in $\mathcal{B}$. If this happens, the method *breaks down*.

So, we will now assume that there are k vectors in $\mathcal{B}$, so that $\mathcal{B}$ spans $\mathbb{R}^k$ and is therefore a basis of $\mathbb{R}^k$, an *eigenbasis* (with respect to A).

Let $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_k)$ be the vectors in $\mathcal{B}$. Each of these vectors is an eigenvector, and we let $\lambda_1, \ldots, \lambda_k$ be the corresponding eigenvalues. This may result in repetitions on the list $\lambda_1, \ldots, \lambda_k$, because some of the eigenspaces may have dimension bigger than 1 (each λ_j appears on the list $(\dim E_j)$-many times).

The method of undetermined coefficients

Supposing that we found an eigenbasis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_k)$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$, the *general solution* to (29) is

$$\vec{x}_n = c_1 \lambda_1^n \vec{v}_1 + \ldots + c_k \lambda_k^n \vec{v}_k \tag{31}$$

The numbers $c_1, \ldots, c_k$ are the (as yet) *undetermined coefficients*. Everything else on the right hand side of (31) is explicitly known at this point (except for n, which is the 'time variable').

The coefficients $c_1, \ldots, c_k$ are determined by the initial state $\vec{x}_0$, if the initial state is given: plug in $n = 0$ into (31):

$$\vec{x}_0 = c_1 \vec{v}_1 + \ldots + c_k \vec{v}_k$$

This gives the inhomogeneous system of linear equations

$$\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \vec{x}_0$$

which we solve for the vector of coefficients $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$

The method of diagonalization

Again, supposing that we found an eigenbasis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_k)$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$, we write down the change of basis matrix

$$P = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix}$$

and the diagonal matrix

$$D = [A]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k \end{pmatrix}$$

and then write down the change of basis formula

$$A = [A]_{\mathcal{E}} = P\ [A]_{\mathcal{B}}\ P^{-1} = P\,D\,P^{-1}$$

which gives a formula for A^n

$$A^n = P\,D^n\,P^{-1}$$

recalling that the powers of a diagonal matrix are straightforward to compute. More explicitly, we get

$$A^n = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1^n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k^n \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix}^{-1}$$

Plugging this into (30) we write down the solution:

$$\vec{x}_n = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1^n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k^n \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix}^{-1} \vec{x}_0$$

The long term behaviour

Still, we assume that we found an eigenbasis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_k)$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_k$.

Let us make a few simplifying assumptions: we will assume that one of the eigenvalues, say λ_1 is larger than the absolute value of all the others

$$\lambda_1 > |\lambda_2| \quad \dots \quad \lambda_1 > |\lambda_k|$$

and that the dimention of the corresponding eigenspace is 1, so that

$$E_1 = \operatorname{span} \vec{v}_1$$

Largest eigenvalue greater than 1. First, let us examine the case that the biggest eigenvalue λ_1 is larger than 1:

$$\lambda_1 > 1$$

We also have to make sure that the initial condition is such that c_1 does not vanish

$$c_1 \neq 0$$

Under these assumptions, the long term behaviour of the dynamical system is determined by λ_1 and E_1. Considering the general solution

$$\vec{x}_n = c_1\lambda_1^n\vec{v}_1 + \ldots + c_k\lambda_k^n\vec{v}_k$$

we see that the term $c_1\lambda_1^n\vec{v}_1$ outgrows all the others, because λ_1^n goes to infinity faster than any of the other terms $\lambda_2^n, \ldots, \lambda_k^n$ (which may not even go to infinity at all). Depending on the numerical values of the entries in $c_1\vec{v}_1$, the individual components of the state vector $\vec{x}_n$ may go to $+\infty$, $-\infty$ or 0. Moreover, the *direction* of the state vector approaches E_1

$$\lim_{n\to\infty} \frac{\vec{x}_n}{||\vec{x}_n||} \in \operatorname{span} \vec{v}_1$$

Largest eigenvalue equal to 1. This time let us assume that $\lambda_1 = 1$. Then all other eigenvalues are less than 1 in absolute value so

$$\lim_{n\to\infty} \lambda_2^n = 0 \qquad \ldots \qquad \lim_{n\to\infty} \lambda_k^n = 0$$

and we have

$$\begin{aligned}
\lim_{n\to\infty} \vec{x}_n &= \lim_{n\to\infty} (c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2 + \ldots + c_k\lambda_k^n\vec{v}_k) \\
&= c_1(\lim_{n\to\infty} \lambda_1^n)\vec{v}_1 + c_2(\lim_{n\to\infty} \lambda_2^n)\vec{v}_2 + \ldots + c_k(\lim_{n\to\infty} \lambda_k^n)\vec{v}_k \\
&= c_1(\lim_{n\to\infty} 1^n)\vec{v}_1 + c_2 \cdot 0 \cdot \vec{v}_2 + \ldots + c_k \cdot 0 \cdot \vec{v}_k \\
&= c_1\vec{v}_1
\end{aligned}$$

so the state vector $\vec{x}_n$ *converges*, and it converges to a point in E_1, namely $c_1\vec{v}_1 \in \operatorname{span} \vec{v}_1$. The limit point on E_1 depends on the initial state (which determines c_1).

Largest eigenvalue less than 1. In this case all the eigenvalues have absolute value less than 1. So $\lim_{n\to\infty} \lambda_j^n = 0$, for all eigenvalues λ_j, and so

$$\lim_{n\to\infty} \vec{x}_n = 0$$

no matter what the initial state is, in the long term the state vector converges to the zero vector, all components of the state vector converge to 0.

1.5 Worked Examples

A 3-dimensional dynamical system

The problem Three mobile telephone companies are competing for customers in Vancouver. Figure 16 shows the yearly patterns according to which they lose customers to each other. Suppose the current market shares of the three companies are: Fido 20%, Telus 40% and Rogers 40%. Give formulas predicting the market shares after n years. What are the market shares in the long run? How many years until the market share of Telus drops below 12%?

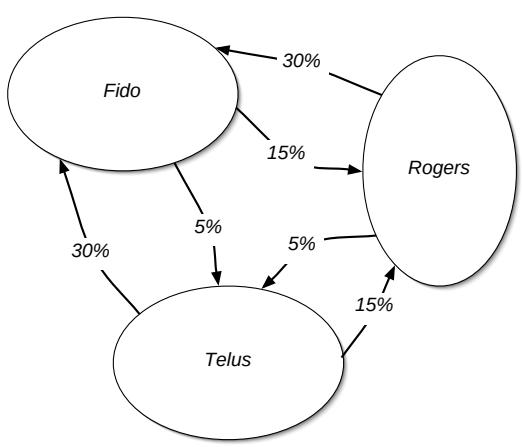


Figure 16: The highly competitive market for cell phone customers in Vancouver

The solution Let us use state vectors $\binom{x}{y}{z}$ where x, y and z are the market shares of Fido, Telus and Rogers, in that order. The migration patterns in

Figure 16 translate into the following three equations

$$\begin{aligned} x_{n+1} &= \tfrac{80}{100}x_n + \tfrac{30}{100}y_n + \tfrac{30}{100}z_n \\ y_{n+1} &= \tfrac{5}{100}x_n + \tfrac{55}{100}y_n + \tfrac{5}{100}z_n \\ z_{n+1} &= \tfrac{15}{100}x_n + \tfrac{15}{100}y_n + \tfrac{65}{100}z_n \end{aligned}$$

or the single matrix equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \frac{1}{100}\begin{pmatrix} 80 & 30 & 30 \\ 5 & 55 & 5 \\ 15 & 15 & 65 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

Thus, the transition matrix for our dynamical system is

$$A = \frac{1}{100}\begin{pmatrix} 80 & 30 & 30 \\ 5 & 55 & 5 \\ 15 & 15 & 65 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 16 & 6 & 6 \\ 1 & 11 & 1 \\ 3 & 3 & 13 \end{pmatrix}$$

The characteristic polynomial is

$$\begin{aligned} \det(\lambda I_3 - A) &= \det\begin{pmatrix} \lambda - \frac{16}{20} & -\frac{6}{20} & -\frac{6}{20} \\ -\frac{1}{20} & \lambda - \frac{11}{20} & -\frac{1}{20} \\ -\frac{3}{20} & -\frac{3}{20} & \lambda - \frac{13}{20} \end{pmatrix} \\ &= \det\frac{1}{20}\begin{pmatrix} 20\lambda - 16 & -6 & -6 \\ -1 & 20\lambda - 11 & -1 \\ -3 & -3 & 20\lambda - 13 \end{pmatrix} \\ &= \frac{1}{8000}\det\begin{pmatrix} 20\lambda - 16 & -6 & -6 \\ -1 & 20\lambda - 11 & -1 \\ -3 & -3 & 20\lambda - 13 \end{pmatrix} \end{aligned}$$

We have to be very careful with the overall factor $\frac{1}{20}$. We cannot ignore it. In either expression for the determinant it is present: either in many denominators, or as a factor in front of λ. Let us use the expression with the fractions:

$$\det\begin{pmatrix} \lambda - \frac{16}{20} & -\frac{6}{20} & -\frac{6}{20} \\ -\frac{1}{20} & \lambda - \frac{11}{20} & -\frac{1}{20} \\ -\frac{3}{20} & -\frac{3}{20} & \lambda - \frac{13}{20} \end{pmatrix} =$$

$$= (\lambda - \tfrac{16}{20})(\lambda - \tfrac{11}{20})(\lambda - \tfrac{13}{20}) + \tfrac{6\cdot1\cdot3}{20^3} + \tfrac{6\cdot1\cdot3}{20^3} - \tfrac{1\cdot3}{20^2}(\lambda - \tfrac{16}{20}) - \tfrac{6\cdot1}{20^2}(\lambda - \tfrac{13}{20}) - \tfrac{6\cdot3}{20^2}(\lambda - \tfrac{11}{20})$$

$$= \lambda^3 - \tfrac{16+11+13}{20}\lambda^2 + \tfrac{11\cdot 13+16\cdot 13+16\cdot 11}{20^2}\lambda - \tfrac{16\cdot 11\cdot 13}{20^3} + \tfrac{36}{20^3} - \tfrac{3+6+18}{20^2}\lambda + \tfrac{3\cdot 16+6\cdot 13+18\cdot 11}{20^3}$$
$$= \lambda^3 - 2\lambda^2 + \tfrac{25}{20}\lambda - \tfrac{5}{20}$$
$$= \lambda^3 - 2\lambda^2 + \tfrac{5}{4}\lambda - \tfrac{1}{4}$$

In general, it is not so easy to find the roots of a cubic polynomial, but in this case we know that one of the eigenvalues will be 1, because we expect there to be a state at which the system stabilizes (no customers are lost or gained overall), which means there will be fixed points, and fixed points are eigenvectors with eigenvalue 1. And, indeed, a quick calculation shows that we get 0, if we plug in $\lambda = 1$ into $\lambda^3 - 2\lambda^2 + \frac{5}{4}\lambda - \frac{1}{4}$. To find the other eigenvalues, we factor out the factor $\lambda - 1$ from the characteristic polynomial: we do (long) polynomial division of $\lambda^3 - 2\lambda^2 + \frac{5}{4}\lambda - \frac{1}{4}$ by $\lambda - 1$ to get $\lambda^2 - \lambda + \frac{1}{4}$, in other words

$$\lambda^3 - 2\lambda^2 + \tfrac{5}{4}\lambda - \tfrac{1}{4} = (\lambda - 1)(\lambda^2 - \lambda + \tfrac{1}{4})$$

Now we are reduced to solving the quadratic $\lambda^2 - \lambda + \frac{1}{4}$, which is easily done, and we obtain only one solution $\lambda = \frac{1}{2}$. So the characteristic polynomial factors as

$$\det(\lambda I_3 - A) = \lambda^3 - 2\lambda^2 + \tfrac{5}{4}\lambda - \tfrac{1}{4} = (\lambda - 1)(\lambda - \tfrac{1}{2})^2$$

and so the two eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$. We follow the usual convention that the largest eigenvalue comes first.

The eigenspace of $\lambda_1 = 1$ is the null space of the matrix

$$\begin{pmatrix} 1 - \frac{16}{20} & -\frac{6}{20} & -\frac{6}{20} \\ -\frac{1}{20} & 1 - \frac{11}{20} & -\frac{1}{20} \\ -\frac{3}{20} & -\frac{3}{20} & 1 - \frac{13}{20} \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & -6 & -6 \\ -1 & 9 & -1 \\ -3 & -3 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

with one free variable z and general solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ \frac{1}{3}z \\ z \end{pmatrix} = z \begin{pmatrix} 2 \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

As basis for the eigenspace E_1 of the eigenvalue $\lambda_1 = 1$, let us take $\vec{v}_1 = \left(\begin{smallmatrix} 6 \\ 1 \\ 3 \end{smallmatrix}\right)$, so as to avoid unsightly fractions.

The space E_1 is the space of fixed points. We have found that it is one-dimensional. We also know that all other eigenvalues are smaller than $\lambda_1 = 1$

in absolute value, so we know that the state vector will converge to a point in E_1. So the limiting market share vector is $c\left(\begin{smallmatrix}6\\1\\3\end{smallmatrix}\right)$, for some scalar c. Since the total market share is 1, we see that c must be $c = \frac{1}{10}$, and the limiting market shares are $x_\infty = \frac{6}{10}$, $y_\infty = \frac{1}{10}$, $z_\infty = \frac{3}{10}$, or Fido 60%, Telus 10% and Rogers 30%. This answers the second question.

The eigenspace of $\lambda_2 = \frac{1}{2}$ is the null space of the matrix

$$\begin{pmatrix} \frac{1}{2} - \frac{16}{20} & -\frac{6}{20} & -\frac{6}{20} \\ -\frac{1}{20} & \frac{1}{2} - \frac{11}{20} & -\frac{1}{20} \\ -\frac{3}{20} & -\frac{3}{20} & \frac{1}{2} - \frac{13}{20} \end{pmatrix} \longrightarrow \begin{pmatrix} -6 & -6 & -6 \\ -1 & -1 & -1 \\ -3 & -3 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with two free variables y and z and general solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

We see that the second eigenspace E_2 is 2-dimensional. As a basis for E_2 we may take the two vectors $\vec{v}_2 = \left(\begin{smallmatrix}-1\\1\\0\end{smallmatrix}\right)$ and $\vec{v}_3 = \left(\begin{smallmatrix}-1\\0\\1\end{smallmatrix}\right)$.

Since the dimensions of the eigenspaces add up to 3, the eigenvalue method works. We have the eigenbasis

$$\vec{v}_1 = \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

with corresponding eigenvalues

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{1}{2} \qquad \lambda_3 = \frac{1}{2}$$

We have now started to list the second eigenvalue twice, because its eigenspace E_2 is 2-dimensional.

Since we are given an explicit initial vector

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \frac{2}{10} \\ \frac{4}{10} \\ \frac{4}{10} \end{pmatrix} \tag{32}$$

let us use the method of undetermined coefficients to solve the dynamical system. The solution is

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_3^n \vec{v}_2 + c_3 \lambda_3^n \vec{v}_3$$

$$= c_1 1^n \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} + c_2 (\tfrac{1}{2})^n \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 (\tfrac{1}{2})^n \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{33}$$

For $n = 0$, we get

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = c_1 \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Plugging in (32) we obtain

$$c_1 \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{10} \\ \frac{4}{10} \\ \frac{4}{10} \end{pmatrix}$$

Or equivalently,

$$\begin{pmatrix} 6 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{10} \\ \frac{4}{10} \\ \frac{4}{10} \end{pmatrix}$$

We use Gaussian elimination on the augmented coefficient matrix

$$\left(\begin{array}{ccc|c} 6 & -1 & -1 & \frac{2}{10} \\ 1 & 1 & 0 & \frac{4}{10} \\ 3 & 0 & 1 & \frac{4}{10} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{10} \\ 0 & 1 & 0 & \frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{10} \end{array}\right)$$

to obtain the unique solution

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \tfrac{1}{10} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

which we plug into (33) to get the final solution

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} &= \tfrac{1}{10} \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} + \tfrac{3}{10} (\tfrac{1}{2})^n \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \tfrac{1}{10} (\tfrac{1}{2})^n \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= \tfrac{1}{10} \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} + \tfrac{1}{10} (\tfrac{1}{2})^n \left(3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right) \end{aligned}$$

$$= \tfrac{1}{10}\begin{pmatrix}6\\1\\3\end{pmatrix} + \tfrac{1}{10}(\tfrac{1}{2})^n\begin{pmatrix}-4\\3\\1\end{pmatrix}$$

or in terms of the individual market shares:

$$\begin{aligned} x_n &= \tfrac{6}{10} - \tfrac{4}{10}(\tfrac{1}{2})^n \\ y_n &= \tfrac{1}{10} + \tfrac{3}{10}(\tfrac{1}{2})^n \\ z_n &= \tfrac{3}{10} + \tfrac{1}{10}(\tfrac{1}{2})^n \end{aligned}$$

These are the formulas predicting the market shares after n years. (A good idea is to check at this point that for $n = 0$ we do get the initial condition, and for $n \to \infty$ we get the limiting market shares.)

Note that y_n starts out at $\frac{4}{10}$ and converges to $\frac{1}{10}$ and decreases in a monotone fashion, so there will be a point when y_n drops below 12%. To find out for what value of n the Telus market share y_n drops below 12% we need to solve the equation $\frac{12}{100} = y_n$ or

$$\tfrac{12}{100} = \tfrac{1}{10} + \tfrac{3}{10}(\tfrac{1}{2})^n$$

for n. Isolate the term involving n on one side:

$$\tfrac{3}{10}(\tfrac{1}{2})^n = \tfrac{2}{100}$$

simplify slightly:

$$15(\tfrac{1}{2})^n = 1$$

Take the logarithm on both sides:

$$\ln(15) - n\ln(2) = 0$$

and solve for n:

$$n = \frac{\ln(15)}{\ln(2)} \approx 3.9$$

So after $n = 4$ years, the Telus market share y_4 will have dropped below 12%.

The powers of a matrix

The problem. Find a formula for A^n, where

$$A = \frac{1}{3}\begin{pmatrix}4 & 2 & 2\\8 & 19 & 4\\-10 & -20 & -5\end{pmatrix}$$

The solution. We will diagonalize A. The characteristic polynomial of A is

$$\det(\lambda I_3 - A) = \det \begin{pmatrix} \lambda - \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{8}{3} & \lambda - \frac{19}{3} & -\frac{4}{3} \\ \frac{10}{3} & \frac{20}{3} & \lambda + \frac{5}{3} \end{pmatrix}$$

$$\begin{aligned} &= (\lambda - \tfrac{4}{3})(\lambda - \tfrac{19}{3})(\lambda + \tfrac{5}{3}) + \tfrac{80}{27} + \tfrac{320}{27} + \tfrac{80}{9}(\lambda - \tfrac{4}{3}) - \tfrac{16}{9}(\lambda + \tfrac{5}{3}) + \tfrac{20}{9}(\lambda - \tfrac{19}{3}) \\ &= \lambda^3 - 6\lambda^2 - \tfrac{39}{9}\lambda + \tfrac{380}{27} + \tfrac{400}{27} + \tfrac{84}{9}\lambda - \tfrac{780}{27} \\ &= \lambda^3 - 6\lambda^2 + 5\lambda \end{aligned}$$

Again, we are lucky that we can find the roots of the cubic: $\lambda = 0$ is an obvious root, which we can easily factor out, and the remaining quadratic is also easy to factor::

$$\det(\lambda I_3 - A) = \lambda^3 - 6\lambda^2 + 5\lambda = \lambda(\lambda^2 - 6\lambda + 5) = \lambda(\lambda - 1)(\lambda - 5)$$

We read off the three eigenvalues $\lambda_1 = 5$, $\lambda_2 = 1$ and $\lambda_3 = 0$, keeping with the convention to take the largest one first.

Now is a good moment to pause and notice that we found 3 different eigenvalues. We will be able to find an eigenvector for each. These 3 eigenvectors will have to be linearly independent, by Fact 1 on Page 40. So they will give us an eigenbasis and so the method of eigenvalues will work.

The eigenspace E_1 corresponding to the eigenvalue $\lambda_1 = 5$ is the null space of the matrix

$$\begin{pmatrix} 5 - \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{8}{3} & 5 - \frac{19}{3} & -\frac{4}{3} \\ \frac{10}{3} & \frac{20}{3} & 5 + \frac{5}{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 11 & -2 & -2 \\ -8 & -4 & -4 \\ 10 & 20 & 20 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

with one free variable z, which we set equal to 1 to obtain the basis

$$\vec{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

of E_1.

The eigenspace E_2 corresponding to the eigenvalue $\lambda_2 = 1$ is the null space of the matrix

$$\begin{pmatrix} 1 - \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{8}{3} & 1 - \frac{19}{3} & -\frac{4}{3} \\ \frac{10}{3} & \frac{20}{3} & 1 + \frac{5}{3} \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -2 & -2 \\ -8 & -16 & -4 \\ 10 & 20 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

with one free variable y, which we set equal to 1 to obtain the basis

$$\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

of E_2.

The eigenspace E_3 corresponding to the eigenvalue $\lambda_3 = 0$ is the null space of the matrix

$$\begin{pmatrix} 0 - \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{8}{3} & 0 - \frac{19}{3} & -\frac{4}{3} \\ \frac{10}{3} & \frac{20}{3} & 0 + \frac{5}{3} \end{pmatrix} \longrightarrow \begin{pmatrix} -4 & -2 & -2 \\ -8 & -19 & -4 \\ 10 & 20 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with one free variable z, which we set equal to 2 to obtain the basis

$$\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

of E_3.

Thus, we have found an eigenbasis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ of $\mathbb{R}^3$. The change-of-coordinates-matrix is

$$P = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

We find its inverse:

$$\left(\begin{array}{ccc|ccc} 0 & -2 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{array}\right)$$

so

$$P^{-1} = \frac{1}{3} \begin{pmatrix} -2 & -4 & -1 \\ -2 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

Now we have the ingredients for the formula

$$\boxed{\boxed{A^n = P\, D^n\, P^{-1}}}$$

where D is the diagonal matrix with the eigenvalues on the diagonal. We plug in our matrices and we get

$$\begin{aligned}
A^n &= P\,D^n\,P^{-1} \\
&= \begin{pmatrix} 0 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 5^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 0^n \end{pmatrix} \frac{1}{3} \begin{pmatrix} -2 & -4 & -1 \\ -2 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 0 & -2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 5^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & -4 & -1 \\ -2 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 0 & -2 & 0 \\ -5^n & 1 & 0 \\ 5^n & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & -4 & -1 \\ -2 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 4 & 2 & 2 \\ 2 \cdot 5^n - 2 & 4 \cdot 5^n - 1 & 5^n - 1 \\ -2 \cdot 5^n & -4 \cdot 5^n & -5^n \end{pmatrix}
\end{aligned}$$

which is the answer to the question. Recalling the original value of A, we can rewrite our answer as

$$\boxed{A^n = (\tfrac{1}{3})^n \begin{pmatrix} 4 & 2 & 2 \\ 8 & 19 & 4 \\ -10 & -20 & -5 \end{pmatrix}^n = \frac{1}{3} \begin{pmatrix} 4 & 2 & 2 \\ 2 \cdot 5^n - 2 & 4 \cdot 5^n - 1 & 5^n - 1 \\ -2 \cdot 5^n & -4 \cdot 5^n & -5^n \end{pmatrix}}$$

But we should be careful to specify that this formula only holds for $n \geq 1$. This is because only for $n \geq 1$ is it true that $0^n = 0$, which we used in our above calculation. For $n = 0$, we have $0^n = 1$, and so our above calculation is wrong. The correct answer for this case is $A^0 = I_3$, which we know anyway without calculation. Because 0 is an eigenvalue of A, the matrix A is singular (its null space is E_3, which we calculated above, and saw to be non-trivial), and so A does not have an inverse. Therefore A^{-1} and all other negative powers of A do not exist. So A^n, for $n < 0$ is *not defined*.

In the case of a matrix where all eigenvalues are non-zero, you will get a formula which applies for all integers n, positive and negative.

We can also rewrite our answer without fractions, but this will give us

the formula for the n-th power of a different matrix, not A:

$$\begin{pmatrix} 4 & 2 & 2 \\ 8 & 19 & 4 \\ -10 & -20 & -5 \end{pmatrix}^n = 3^{n-1} \begin{pmatrix} 4 & 2 & 2 \\ 2 \cdot 5^n - 2 & 4 \cdot 5^n - 1 & 5^n - 1 \\ -2 \cdot 5^n & -4 \cdot 5^n & -5^n \end{pmatrix}$$

It is always a good idea to check that our answer makes sense. In general, good cases to check are $n = 0$ (where we should be getting the identity matrix) and $n = 1$ (where we should be getting the matrix we started with). In our case, because the matrix is singular, we can only check our result against $n = 1$, or other positive powers.

1.6 More on Markov processes

Definition 2 A **stochastic matrix** is a square matrix A, such that

(i) every entry of A is non-negative,

(ii) every column of A adds to 1.

A discrete dynamical system with a stochastic transition matrix is called a *Markov process*, or a *Markov chain*.

Both the Vancouver-Richmand population problem, and the cell phone customer problem were described by a stochastic matrix. If we think of population dynamics, as in these two examples, the fact that the overall population stays constant implies that the transition matrix is stochastic. This is because the i-th column of the transition matrix describes the outflow from the i-th city or phone company. The total migration out of every city or phone company is 1.

For every stochastic matrix we can draw a graph such as Figure 16 or Figure 1, and for every such graph, we can find the corresponding stochastic matrix. The two representations of the dynamical system are entirely equivalent.

Theorem 3 *Let A be a stochastic matrix. Then 1 is an eigenvalue of A. If λ is an eigenvalue of A (real or complex), then $|\lambda| \leq 1$.*

Proof. Let us prove that 1 is an eigenvalue. Suppose B is a matrix whose *rows* add to 1. Then you can check that the vector $\vec{v}$, all of whose entries are 1 is an eigenvector of B, with eigenvalue 1. So 1 is a root of the characteristic

polynomial of B. But a matrix and its transpose have the same characteristic polynomial, and hence the same eigenvalues.

The fact about the other eigenvalues is also not hard to prove, but it is also intuitive: if there were an eigenvalue λ, with $|\lambda| > 1$, the system would exhibit long-term overall growth behaviour, which is impossible, because the overall population stays constant, for a Markov process. $\square$

Theorem 4 (Perrin-Frobenius) *Suppose all entries of the stochastic matrix A are positive, i.e., non-zero. Then all eigenvalues λ other than 1 satisfy $|\lambda| < 1$, and 1 is a simple eigenvalue, i.e., its algebraic multiplicity is equal to 1.*

Moreover, there exists a fixed vector all of whose components are positive.

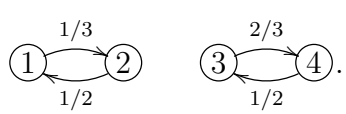
This theorem says that if a Markov process has a *positive* transition matrix (i.e., all entries are positive) then, no matter what the initial condition is, the system will converge to a steady state, and the steady state is unique up to rescaling. (The correct scaling factor can be determined from the overall population.) The 'moreover' ensures that none of the individual populations will die out, or become negative. The solution is physically viable.

Unfortunately, this theorem does not hold if the stochastic matrix has zero entries.

Example 5 Consider the stochastic matrix

$$A = \begin{pmatrix} 2/3 & 1/2 & 0 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 1/2 \\ 0 & 0 & 2/3 & 1/2 \end{pmatrix}.$$

The associated graph is



No matter the initial condition, the system will converge to an equilibrium point. But the equilibrium will depend on the initial condition. The eigenspace E_1 is two-dimensional.

$$\begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix}$$

are two linearly independent fixed vectors.

Thinking in terms of cities and their populations, we can group Cities 1 and 2 into Country A, and Cities 2 and 3 into Country B. Then the equilibrium point will depend on how the initial population is distributed between the two countries, because there is no interaction between the two countries.

We see that the fact that the graph is not connected is responsible for the ambiguity in the equilibrium state.

Example 6 Consider the stochastic matrix

$$B = \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 5/12 & 0 \\ 0 & 1/3 & 1 \end{pmatrix} .$$

The associated graph is

$$1 \xleftarrow{1/4} 2 \xrightarrow{1/3} 3 .$$

This time the graph is connected, but there are still two linearly independent equilibria, namely $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$.

All that happens in this Markov process is that City 2 gets depleted over time, and its popultian moves to Cities 1 and 2. The limiting distribution depends on the initial condition.

The problem with this example is that there is no way for any popultion to move from City 1 to City 3, or from City 3 to City 1.

Because of these two examples, we introduct the following notion.

Definition 7 The stochastic matrix A is called **irreducible**, if the associated graph is **strongly connected**, which means that you can get from each vertex to every other vertex by following along arrows with strictly positive percentages on them.

So the graph in Example 6 is not strongly connected, the stochastic matrix B is not irreducible. Of course, every *positive* stochastic matrix as in the Perrin-Frobenius theorem is strongly connected, because there is an arrow from every vertex to every other vertex.

Theorem 8 *If A is an irreducible stochastic matrix, then the eigenvalue 1 is a simple eigenvalue (it is a simple root of the characteristic polynomial).*

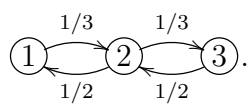
In particular, the eigenspace E_1 is one-dimensional, so the steady state is unique up to scaling.

Moreover, there exists a steady state where all entries are positive.

Example 9 Here is a simple example of a stochastic matrix which is not positive, but still irreducible.

$$C = \begin{pmatrix} 2/3 & 1/2 & 0 \\ 1/3 & 1/6 & 1/2 \\ 0 & 1/3 & 1/2 \end{pmatrix},$$

with associated graph



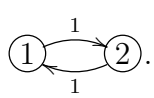
The space of fixed point is span $\begin{pmatrix} 9 \\ 6 \\ 4 \end{pmatrix}$, which is indeed one-dimensional.

Irreducible stochastic matrices can still exhibit undesirable behaviour.

Example 10 Consider the reflection matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is a stochastic matrix. The corresponding graph is



In fact, A is an irreducible stochastic matrix, you can get from Vertex 1 to Vertex 2 and from Vertex 2 to Vertex 1 by following along one of the arrows in the graph. So we know, by Theorem 8 that there is a unique steady state, up to scaling.

At each iteration, the populations swap completely. Everyone in City 1 moves to City 2, and everyone in City 2 moves to City 1. If, initially, the populations of the two cities are equal, they will remain equal, and we are in the (unique) steady state. If we start with an unbalanced initial condition, the population of each city will oscillate, with a period of 2 (all odd years

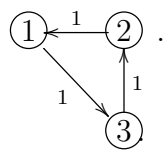
have the same population, and all even years have the same population, everything repeats after two years). This stochastic matrix has a period of 2.

The eigenvalues of A are 1 and -1. The oscillatory behaviour is due to the eigenvalue -1, which is not equal to 1, but has absolute value 1.

Example 11 Consider the 3×3-matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

It is a stochastic matrix. The corresponding graph is



Again, B is irreducible, and the steady state corresponds to equal population in the three cities.

But unless the initial condition is completely balanced, the populations oscillate, this time with a period of 3. In each city, the population number repeats with a cycle of 3 iterations.

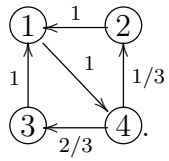
The characteristic polynomial of B is $\lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1)$. As expected $(\lambda - 1)$ occors with multiplicity 1 in this degree 3 polynomial. The two other eigenvalues are complex, and they both have absolute value 1. Again, the oscillatory behavior is due to these eigenvalues which are not 1, but have absolute value 1.

We can see where the oscillatory behaviour in these two examples comes from: the cities can be arranged in a circle, such that all arrows in the circle go around in the same direction, and are all labelled with 1, corresponding to 100% population movement. We have to exclude such graphs, if we want to assure convergence to a steady state.

Example 12 Consider the irreducible stochastic matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 2/3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

whose associated graph looks like this



Again, there is oscillatory behaviour, althout it is a bit less obvious. But we can group the 4 cities into 3 contries, such that we have a cycle of length 3 between countries, instead of cities, exactly like in Example 11. In fact, let City 1 occupy Country A, group Cities 2 and 3 together into Country B, and let City 4 by its own Country C. This corresponds to partitioning the matrix into blocks:

$$C = \left(\begin{array}{c|cc|c} 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 2/3 \\ \hline 1 & 0 & 0 & 0 \end{array}\right),$$

which looks exactly like B if you 'squint your eyes'.

Theorem 13 *Let A be an irreducible stochastic matrix. Suppose that there is no way to group the vertices of the associated graph into groups, and then put the groups into a circle, such that there is 100% population movement in one direction around the circle of groups (ignoring what happens inside each group). Then all eigenvalues $\lambda \neq 1$ of A (real and complex) satisfy $|\lambda| < 1$.*

So in this case, no matter what the initial condition is, the Markov process associated to A will converge to a steady state, this steady state is unique up to scaling, and all of its components are positive.

So under the conditions of this last theorem, we only need to solve the fixed point equation to determine the long-term behaviour of the Markov process. The fixed point equation will give us the proportional populations, and we can determine the exact equilibrium populations from the overall population. Moreover, we can be sure that none of the individual populations die out or become negative.

Remark 14 In this course, you should assume that all Markov processes you encounter have a one-dimensional fixed space, and exhibit no oscillatory behaviour. You will not be expected to check for irreducibitliy of the stochastic transition matrix, or verify that there is no oscillatory behaviour.

1.7 Exercises

Exercise 1.1 Without calculating the characteristic polynomial of A, determine which of $\lambda = 0, 1, 2, 3$ are eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Exercise 1.2 Find all equilibrium points of the dynamical system with transition matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.3 Referring to the migration problem between Vancouver and Richmond, find exact formulas for v_n and r_n. The initial conditions are $v_0 = 300$ and $r_0 = 100$, as in the text.

Exercise 1.4 Repeat the analysis of the Fibonacci system, but assume that every adult pair of rabbits produces 2 pairs of offspring every month. (The initial condition is the same: 1 juvenile pair.) In particular, find exact formulas for j_n and a_n, find $\lim_{n\to\infty} \frac{a_n}{j_n}$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

Exercise 1.5 Solve the 2×2 dynamical system

$$\begin{aligned} x_{n+1} &= x_n - y_n \\ y_{n+1} &= 2x_n + 4y_n \\ x_0 &= 2 \\ y_0 &= 5 \end{aligned}$$

This means to find the formulas for x_n and y_n in terms of n.

Exercise 1.6 Find the general solution of the dynamical system

$$\vec{x}_{n+1} = \tfrac{1}{21} \begin{pmatrix} 37 & 10 \\ 15 & 12 \end{pmatrix} \vec{x}_n$$

and sketch the phase portrait. What initial conditions will prevent the system from growing beyond all bounds?

Exercise 1.7 Find a formula for $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)^n$, the n-th power of the transition matrix of the Fibonacci system.

Exercise 1.8 Modify the numbers in the rabbit problem, to make it more realistic. Assume that the likelihood for an adult pair to produce offspring in one month is only 3 out of 5 or 60%. Also, assume that the death rate of adult rabbits is 50%, which means that out of 100 adult pairs of rabbits, only 50 will survive until the next month. Also, the death rate of juvenile rabbits is 1 in 3, so that the likelihood of a juvenile pair of rabbits to grow into an adult pair of rabbits is $\frac{2}{3}$. Determine if the rabbit population will survive, or if it will die out. Find the limiting growth rate of the population. (Hint: you do not need to find the exact solution of the dynamical system, you only need to find the eigenvalues, to solve this problem.)

Assuming the same birth rate and death rate for adult rabbits, what is the smallest survival rate for juveniles which will assure the survival of the rabbit population?

Exercise 1.9 Solve the 3×3 dynamical system with transition matrix

$$A = \frac{1}{3}\begin{pmatrix} 7 & -2 & 2 \\ 3 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and initial condition

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ -5 \end{pmatrix}$$

(Hint: one of the eigenvalues of A is 2.)

Exercise 1.10 Find a formula for A^n, where $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

2 Continuous Dynamical Systems

2.1 Flow Example

Two cubic metres ($2m^3$) of a pollutant are spilled into Upper Lake. Predict the time dependence of the concentration of the pollutant in Lower Lake.

At what point in time does the concentraion of the pollutant in Lower Lake reach its maximal value? Will Lower Town on Lower Lake have to close its beach? (The health board considers a concentration of $0.008\,ppb$ (parts per billion) of this particular chemical to be safe for swimming, but requires beaches to be closed, if the concentration exceeds this value.) If so, for how long will the beach need to be closed?

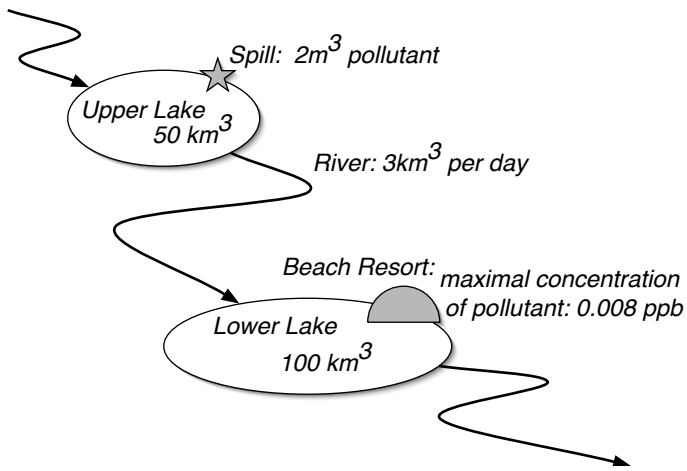


Figure 17: The lay of the land

Upper Lake has a volume of $50\,km^3$, Lower Lake a volume of $100\,km^3$. A river flows into Upper Lake, connects the two lakes and flows out of Lower Lake. The river flows at a rate of $3\,km^3$ per day.

Let us use state vectors $\binom{x}{y}$, where x is the amount of pollutant in Upper Lake and y the amount of pollutant in Lower Lake, both measured in m^3. The state vector will be time dependent.

2.2 Discrete Model

If we model the situation with a discrete dynamical system, with time interval one day, we obtain the following:

$$x_{n+1} = x_n - \tfrac{3}{50}x_n \tag{34}$$

$$y_{n+1} = y_n + \tfrac{3}{50}x_n - \tfrac{3}{100}y_n \tag{35}$$

The amount of pollutant in Upper Lake tomorrow, x_{n+1}, is equal to the amount of pollutant in Upper Lake today, x_n, minus the amount of pollutant

which has flown into Lower Lake, via the river. The concentration of pollutant in Upper Lake today is $\frac{x_n}{50}\frac{m^3}{km^3}$. The amount of water carried by the river to Lower Lake during the day is $3\,km^3$. This water contains

$$3\,km^3\,\frac{x_n}{50}\frac{m^3}{km^3} = \frac{3}{50}x_n\,m^3$$

of the pollutant. So the decrease of pollutant in Upper Lake is equal to $\frac{3}{50}x_n$ (measured in m^3, which are the correct units for the components of the state vector). This justifies Formula (34).

The change in the pollutant in Lower Lake consists of the inflow from Upper Lake, which is $\frac{3}{50}$, minus the outflow, which is $3\frac{y_n}{100}$, because the concentration of the pollutant in Lower Lake is $\frac{y_n}{100}$.

2.3 Refining the Discrete Model

If we look closer at the system described by (34) and (35), we see that it is not as accurate as it might be: when calculating the amount of pollutant flowing out of Upper Lake during a day, we assumed that the concentration of the pollutant in upper lake is $\frac{x_n}{50}$ for the whole day. This does not take into account that the concentration decreases during the day. If $\frac{x_n}{50}$ is correct in the morning, it will not be correct any more in the evening, because pollutant has been flowing out all day.

To get a more accurate decription, we may decide to decrease the time interval to half a day. How would this affect the description of the system?

The main change we will make is the following: instead of writing x_n for the pollutant in Upper Lake after n days, we will write $x(t)$, for the amount of pollutant in Upper Lake at time t. Now we think of t as continuously varying: $t \in \mathbb{R}$. We measure time in days, so $t = 5$ stands for 5 days (which used to be $n = 5$). With this new notation, the above system reads like this:

$$\begin{aligned} x(t+1) &= x(t) - \tfrac{3}{50}x(t) \\ y(t+1) &= y(t) + \tfrac{3}{50}x(t) - \tfrac{3}{100}y(t) \end{aligned}$$

If we change the time interval to half a day, the river carries half as much water in half a day. So the new system is:

$$\begin{aligned} x(t+\tfrac{1}{2}) &= x(t) - \tfrac{1}{2}\tfrac{3}{50}x(t) \\ y(t+\tfrac{1}{2}) &= y(t) + \tfrac{1}{2}\tfrac{3}{50}x(t) - \tfrac{1}{2}\tfrac{3}{100}y(t) \end{aligned}$$

Of course, we could be even more accurate by looking at the system every hour. The day has 24 hours, so if we model the system using hours for our time interval, we get:

$$\begin{aligned} x(t+\tfrac{1}{24}) &= x(t) - \tfrac{1}{24}\tfrac{3}{50}x(t) \\ y(t+\tfrac{1}{24}) &= y(t) + \tfrac{1}{24}\tfrac{3}{50}x(t) - \tfrac{1}{24}\tfrac{3}{100}y(t) \end{aligned}$$

The other change we make is that we look at the *change* that happens during our time interval, instead of the absolute numbers:

$$\begin{aligned} x(t+\tfrac{1}{24}) - x(t) &= -\tfrac{1}{24}\tfrac{3}{50}x(t) \\ y(t+\tfrac{1}{24}) - y(t) &= \tfrac{1}{24}\tfrac{3}{50}x(t) - \tfrac{1}{24}\tfrac{3}{100}y(t) \end{aligned}$$

If we divide by the time interval, we get:

$$\begin{aligned} \frac{x(t+\frac{1}{24}) - x(t)}{\frac{1}{24}} &= -\tfrac{3}{50}x(t) \\ \frac{y(t+\frac{1}{24}) - y(t)}{\frac{1}{24}} &= \tfrac{3}{50}x(t) - \tfrac{3}{100}y(t) \end{aligned}$$

2.4 The continuous model

As we decrease the time interval it gets cumbersome to write larger and larger fractions ($\frac{1}{1440}$ for minutes, $\frac{1}{86400}$ for seconds), so let us write h for the time interval:

$$\begin{aligned} \frac{x(t+h) - x(t)}{h} &= -\tfrac{3}{50}x(t) \\ \frac{y(t+h) - y(t)}{h} &= \tfrac{3}{50}x(t) - \tfrac{3}{100}y(t) \end{aligned}$$

Note that the right hand side does not depend on h at all. If we take the limit $h \to 0$, we get the *instantaneous rates of change* on the left hand side, also known as the *derivatives:*

$$\begin{aligned} \frac{dx}{dt}(t) &= \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} = -\tfrac{3}{50}x(t) \\ \frac{dy}{dt}(t) &= \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} = \tfrac{3}{50}x(t) - \tfrac{3}{100}y(t) \end{aligned}$$

Thus we arrive at the continuous model:

$$\frac{dx}{dt}(t) = -\tfrac{3}{50}x(t) \tag{36}$$

$$\frac{dy}{dt}(t) = \tfrac{3}{50}x(t) - \tfrac{3}{100}y(t) \tag{37}$$

Or, in matrix notation:

$$\begin{pmatrix} \frac{dx}{dt}(t) \\ \frac{dy}{dt}(t) \end{pmatrix} = \begin{pmatrix} -\frac{3}{50} & 0 \\ \frac{3}{50} & -\frac{3}{100} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

If we don't write the dependence on t explicitely, we can write this even shorter as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{50} & 0 \\ \frac{3}{50} & -\frac{3}{100} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Writing $\vec{x}$ for the state vector $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ this becomes

$$\tfrac{d}{dt}\vec{x} = \begin{pmatrix} -\frac{3}{50} & 0 \\ \frac{3}{50} & -\frac{3}{100} \end{pmatrix} \vec{x}$$

The transition matrix is

$$A = \begin{pmatrix} -\frac{3}{50} & 0 \\ \frac{3}{50} & -\frac{3}{100} \end{pmatrix}$$

With this notation the dynamical system becomes:

$$\boxed{\boxed{\tfrac{d}{dt}\vec{x}(t) = A\,\vec{x}(t)}}$$

Often abbreviated to

$$\boxed{\boxed{\vec{x}\,'(t) = A\,\vec{x}(t)}}$$

This is known as a *homogeneous system of linear differential equations.*

2.5 Solving the system of differential equations

One homogeneous linear differential equation

One homogeneous linear differentail eqaution

$$x'(t) = \lambda x(t)$$

is easy to solve. The general solution is

$$x(t) = ce^{\lambda t}$$

for an undetermined constant c, which can be determined from $x(0)$, if $x(0)$ is given.

For example, the *initial value problem*

$$x'(t) = 15x(t) \tag{38}$$
$$x(0) = 3 \tag{39}$$

has the solution

$$x(t) = 3e^{15t} \tag{40}$$

It is easy to check that (40) is a solution of (38) and (39): taking the derivative of (40) gives $x'(t) = 15 \cdot 3e^{15t}$, and plugging (40) back into this gives $x'(t) = 15x(t)$, which is (38). Plugging in 0 for t in (40) gives (39).

The concentration of the pollutant in Upper Lake has nothing to do with what happens in Lower Lake. Equation (36) is a single differential equation for the function $x(t)$, giving the pollutant in Upper Lake. The solution is

$$x(t) = 2e^{-\frac{3}{50}t}$$

because the initial condition was given as $x(0) = 2$: the amount of pollutant spilled into Upper Lake was $2\,m^3$.

Our system of linear differential equations

Suppose the state vector has n components, $\vec{x} \in \mathbb{R}^n$, and the transition matrix A is $n \times n$. If we can find a basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$ consisting of eigenvectors for A, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then the general solution to

$$\boxed{\boxed{\vec{x}\,'(t) = A\,\vec{x}(t)}} \tag{41}$$

is

$$\boxed{\boxed{\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n}} \tag{42}$$

Let us check that (42) is, indeed, a solution to (41). What does (41) say? It says that taking the derivative of the state vector gives the same result as multiplying the state

vector by the transition matrix. So what we are claiming is that if we start with the state vector

$$c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n \tag{43}$$

we get the same result by differentiating, as when we multiply by A. Let us check this. Differentiating (43) we get

$$\lambda_1 c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + \lambda_n c_n e^{\lambda_n t}\vec{v}_n$$

because the c_i are constants, and the eigenvectors $\vec{v}_i$ don't depend on t, either. Multiplying (43) by A we get

$$\begin{aligned} A\big(c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n\big) &= c_1 e^{\lambda_1 t}A\vec{v}_1 + \ldots + c_n e^{\lambda_n t}A\vec{v}_n \\ &= c_1 e^{\lambda_1 t}\lambda_1\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\lambda_n\vec{v}_n \end{aligned}$$

because $A\vec{v}_i = \lambda_i\vec{v}_i$, by the eigenvalue property. This is, indeed, the same result we got by differentiating.

In our example, the transition matrix is

$$A = \begin{pmatrix} -\frac{3}{50} & 0 \\ \frac{3}{50} & -\frac{3}{100} \end{pmatrix}$$

It is lower triangular, so the eigenvalues are on the diagonal:

$$\lambda_1 = -\tfrac{3}{100} \qquad \lambda_2 = -\tfrac{3}{50}$$

we have listed $-\frac{3}{100}$ first, because it is larger than $-\frac{3}{50}$. Find the corresponding eigenspaces: E_1, the eigenspace of λ_1, is the null space of

$$\begin{pmatrix} -\frac{3}{100}+\frac{3}{50} & 0 \\ -\frac{3}{50} & -\frac{3}{100}+\frac{3}{100} \end{pmatrix} = \begin{pmatrix} \frac{3}{100} & 0 \\ -\frac{3}{50} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so

$$E_1 = \operatorname{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For λ_2 we get

$$\begin{pmatrix} -\frac{3}{50}+\frac{3}{50} & 0 \\ -\frac{3}{50} & -\frac{3}{50}+\frac{3}{100} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{3}{50} & -\frac{3}{100} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

so

$$E_2 = \operatorname{span}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

So our eigenbasis is

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Plugging these values into (42) we get

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-\frac{3}{100}t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{-\frac{3}{50}t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

To compute c_1 and c_2, plug in $t = 0$:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The initial values are $x(0) = 2$ and $y(0) = 0$, so this gives the inhomogeneous system for $\vec{c}$:

$$c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Solving it

$$\begin{pmatrix} 0 & 1 & | & 2 \\ 1 & -2 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & 2 \end{pmatrix}$$

gives $c_1 = 4$ and $c_2 = 2$. Our final solution is

$$\boxed{\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 4e^{-\frac{3}{100}t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2e^{-\frac{3}{50}t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}}} \tag{44}$$

The formula for the pollutant in Upper Lake

$$x(t) = 2e^{-\frac{3}{50}t}$$

we already knew. The formula for the amount of pollutant in Lower Lake is

$$y(t) = 4e^{-\frac{3}{100}t} - 4e^{-\frac{3}{50}t}$$

If we consider this formula for $y(t)$, we notice several things:

(i) $y(0) = 0$,

(ii) $y(t) > 0$, for all $t > 0$, because $e^{-\frac{3}{100}t} > e^{-\frac{3}{50}t}$, because $-\frac{3}{100} > -\frac{3}{50}$.

(iii) as $t \to \infty$ both summands of $y(t)$ approach 0, so $\lim_{t\to\infty} y(t) = 0$.

By these observations, it is clear that $y(t)$ must reach a maximum for some positive point in time t. To find it, let us differentiate $y(t)$

$$y'(t) = -\tfrac{4\cdot 3}{100}e^{-\frac{3}{100}t} + \tfrac{4\cdot 3}{50}e^{-\frac{3}{50}t}$$

and set $y'(t) = 0$

$$\tfrac{4\cdot 3}{100}e^{-\frac{3}{100}t} = \tfrac{4\cdot 3}{50}e^{-\frac{3}{50}t}$$

which gives

$$e^{-\frac{3}{100}t} = 2e^{-\frac{3}{50}t}$$

Take the logarithm:

$$-\tfrac{3}{100}t = \ln 2 - \tfrac{3}{50}t$$

this gives

$$\tfrac{3}{100}t = \ln 2$$

or

$$t_{\max} = \frac{100\ln 2}{3} \approx 23.1$$

So the maximum concentration of the pollutant in Lower Lake is reached after about 23 days. How much pollutant is in Lower Lake at this point in time? This value is given by $y(t_{\max})$ and is equal to

$$\begin{aligned} y(t_{\max}) &= 4e^{-\frac{3}{100}t_{\max}} - 4e^{-\frac{3}{50}t_{\max}} \\ &= 4e^{-\frac{3}{100}\frac{100\ln 2}{3}} - 4e^{-\frac{3}{50}\frac{100\ln 2}{3}} \\ &= 4e^{-\ln 2} - 4e^{-2\ln 2} \\ &= 4\cdot 2^{-1} - 4\cdot 2^{-2} \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

So the maximum amount of pollutant in Lower Lake is $1\,m^3$. This corresponds to a concentration of $\frac{1}{100}\frac{m^3}{km^3}$, or $0.01\,ppb$, because there are 1 billion=1000^3 cubic metres in 1 cubic kilometre. So the maximal concentration of the pollutant will exceed the maximal allowable value of $0.008\,ppb$. The beach will certainly need to be closed.

The maximal allowable concentration $0.008\,ppb$ corresponds to an amount of $y = 0.8\,m^3$ of the pollutant in lower lake. To find out exactly on which days the beach needs to be closed we would have to solve the equation

$$y(t) = 0.8$$

or

$$4e^{-\frac{3}{100}t} - 4e^{-\frac{3}{50}t} = 0.8$$

This is not possible with elementary methods. We are reduced to 'trial and error'. We use a computer to calculate values of $y(t)$:

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
y	.11	.22	.32	.40	.48	.55	.61	.67	.72	.76	.81	.84	.88	.90
t	15	16	17	18	19	20	21	22	23	24	25	26	27	
y	.92	.95	.96	.97	.98	1.00	.99	1.00	1.00	1.00	.998	.990	.988	
t	27	28	29	30	31	32	33	34	35	36	37	38		
y	.988	.986	.976	.970	.956	.942	.938	.920	.912	.900	.884	.872		
t	38	39	40	41	42	43	44	45	46	47	48	49		
y	.872	.855	.837	.828	.818	.797	.784	.771	.757	.738	.724	.708		

We see that the beach on Lower Lake will have to be closed from day 11 until day 42, i.e., for 32 days in total. The beach resort in Lower Town has 10 days to prepare for this emergency.

Of course, even this continuous model is very far from perfect. We have entirely neglected the time it takes for the pollutant to get evenly distributed throughout the lakes, which may be considerable, in real life. Also, we have ignored the time it takes the water to flow down the river from Upper Lake to Lower Lake.

2.6 Phase Portrait

Both eigenvalues of the transition matrix are negative. If $\lambda < 0$, then as $t \to +\infty$, $e^{\lambda t} \to 0$. This implies that both terms in the formula for $\vec{x}(t)$ decay in the long term. So the state vector gets smaller and smaller, it approaches $\vec{0}$:

$$\lim_{t \to +\infty} = \vec{0}$$

The origin is an *attractor* for all trajectories.

As $t \to +\infty$, the term with the larger eigenvalue takes over, so the slope of the state vector approaches the slope of $\vec{v}_1 = \binom{0}{1}$. So as the trajectories approach the origin, they sqeeze closer and closer to E_1, getting farther away from E_2 in the process.

As $t \to -\infty$, the term with the smaller eigenvalue takes over, so the slope of the state vector approaches the slope of $\vec{v}_2 = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$. So if we follow the trajectories backwards, they align themselves more and more with E_2 (even though they do not get closer to E_2). This can be seen in Figure 18.

2.7 Exercises

Exercise 2.1 Solve the discrete dynamical system described by (34) and (35) with the initial condition given in the text. Find explicitel formulas for x_n and y_n. Compare your answer with the solution of the continuous model (44).

Exercise 2.2 Solve the initial value problem

$$\begin{aligned} x'(t) &= 10x(t) + 4y(t) \\ y'(t) &= \quad\ x(t) + 10y(t) \\ x(0) &= 4 \\ y(0) &= 0 \end{aligned}$$

Exercise 2.3 Consider the same dynamical system as in the text (with the same initial condition), except that the sizes of the two lakes are interchanged. Repeat the analysis in the text. Solve the continuous dynamical system, and sketch the phase portrait. Is the beach safe?

Exercise 2.4 Consider the same dynamical system as in the text, except that there is a third lake, Middle Lake, in between Upper Lake and Lower Lake. The volume of Middle Lake is $150\,km^3$. Solve the initial value problem.

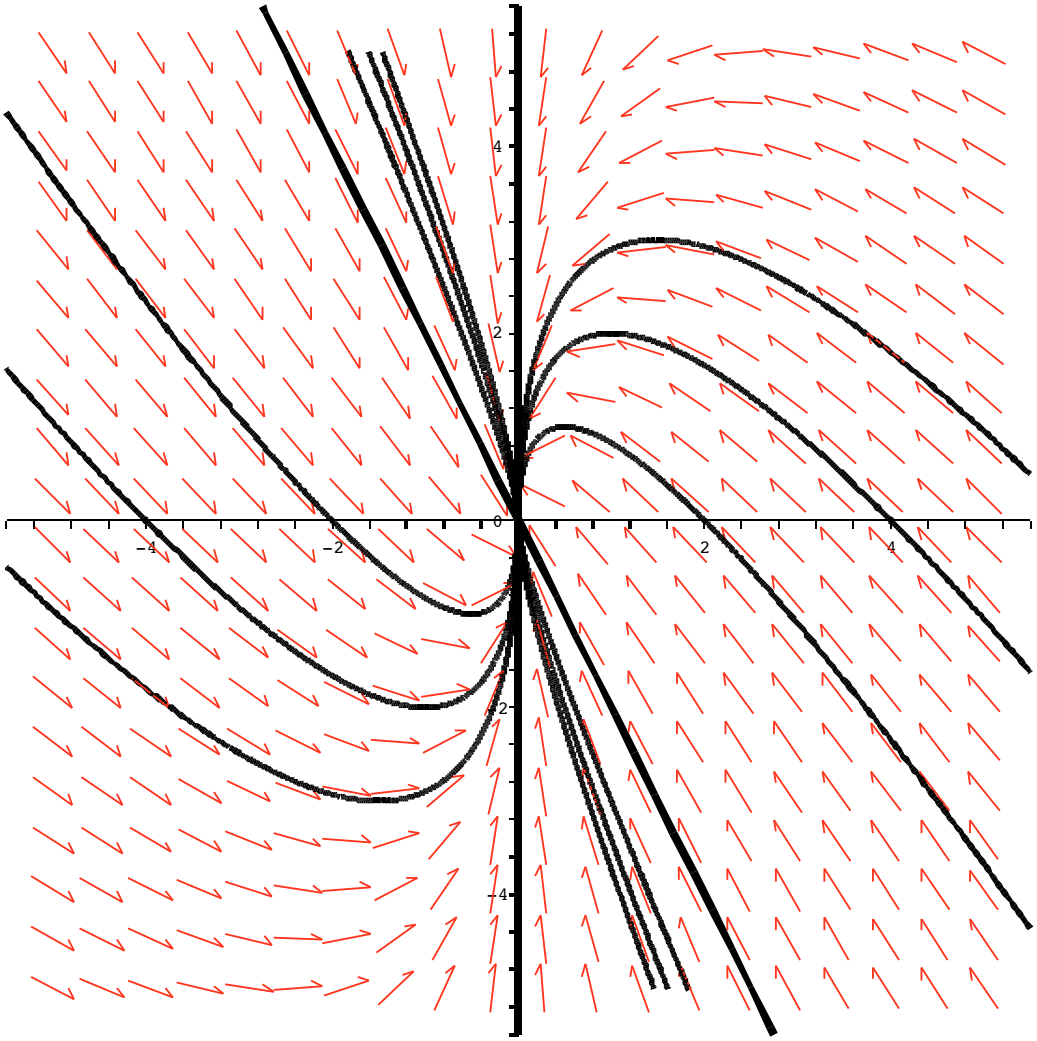


Figure 18: The phase portrait of the lake system