Recent advances in the MMP, after Shokurov, II

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- To achieve this birational classification, we propose to use the MMP.

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Conjecture. *(Existence) Suppose that* $K_X + \Delta$ *is log terminal. Let* $\pi\colon X \longrightarrow Y$ *be a small extremal contraction.*
Then the flip of π *exists.*

Conjecture. *(Termination) There is no infinite sequence of log terminal flips.*

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- The first step of the proof, is to reduce the dimension by one. Therefore we are free to use the MMP.
- Many of the ideas in his paper will probably influence other work in higher dimensional geometry.

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- 110, in a manuscript with 245 pages.

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- (Vanishing) The simplest form is Kodaira vanishing which states that if X is smooth and L is an ample line bundle, then $H^i(K_X + L) = 0$, for $i > 0$.
- Both of these results have far reaching generalisations, whose form dictates the main definitions of the subject.

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- Pick $a > 0$ so that $K_S + aH$ is ample.
- Then pick b so that $(K_S + aH + bE) \cdot E = 0$. Note that $b > 0$ (in fact typically b is very large).
- Now we consider the rational map given by $|mD|$, for $m >> 0$ and sufficiently divisible.

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and $G + (m-1)D$ is ample.
- So by Kawamata-Viehweg Vanishing

$$H^1(S, \mathcal{O}_S(mD - E)) = H^1(S, \mathcal{O}_S(K_S + G + (m-1)D)) = 0$$

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- B is ample, so we have the start of an induction.
- By vanishing, the map

$$H^0(S, \mathcal{O}_S(mD)) \longrightarrow H^0(E, \mathcal{O}_E(mD))$$

is surjective. Thus $|mD|$ is base point free and the resulting map $S \longrightarrow T$ contracts E.

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- Observe that if we set $G' = \pi_* G$, then G' has high multiplicity along p, the image of E (that is b is large).
- In general, we manufacture a divisor E by picking a point $x \in X$ and then pick H with high multiplicity at x.
- Next resolve singularities $\tilde{X} \longrightarrow X$ and restrict to an exceptional divisor E, whose centre has high multiplicity w.r.t H (strictly speaking a log canonical centre of $K_X + H$).

Singularities in the MMP

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- Moreover if $K_X + S + B$ is plt then $K_S + D$ is klt.

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- If we take a cover with appropriate ramification, then we can eliminate any component with coefficient less than one.
- (Kawamata-Viehweg vanishing) Suppose that $K_X + \Delta$ is klt and L is a line bundle such that $L - (K_X + \Delta)$ is big and nef. Then, for $i > 0$,

$$H^i(X, L) = 0.$$

General Set up

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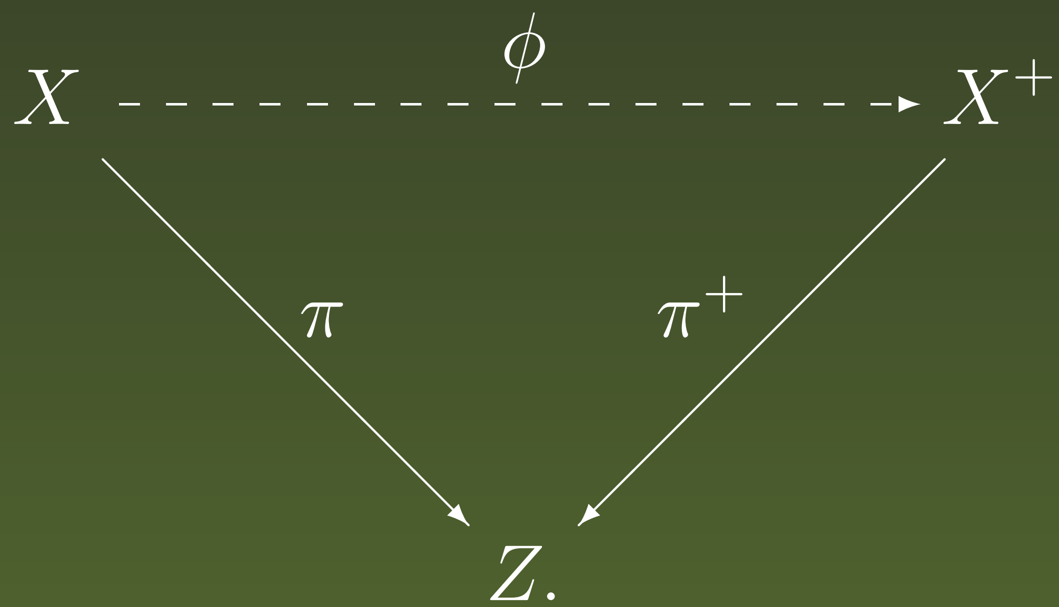

Reduction to pl Flips

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Theorem. *Pl flips terminate in dimension four.*

Finite Generation

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- Recall that the flip exists iff

$$R = R(X, D) = \bigoplus_n H^0(X, \pi_* \mathcal{O}_X(nD)),$$

is finitely generated.

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- Thus R is finitely generated iff $\operatorname{res}_S R$ is finitely generated.

Function algebras

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It is easy to see that a restricted algebra is a bounded function algebra.

b-divisors

Definition. *A b-divisor on a normal variety is an element:*

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where the limit runs over all proper birational maps $Y \longrightarrow X$.

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An infinite formal sum of valuations $\sum a_E E$. In this case the trace is

$$D_Y = \sum_{E \text{ is a divisor on } Y} a_E E$$

Examples of b-divisors

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for any model $f \colon Y \longrightarrow X$.

- Suppose we have a pair (X, Δ). The discrepancy b-divisor $A = A(X, \Delta)$ is defined by

$$K_Y = f^*(K_X + \Delta) + A(X, \Delta)_Y.$$

Linear equivalence of $\mathbf{b}$-divisors

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Here is a key example. Let $X = \mathbb{P}^2$. Pick a point $p \in \mathbb{P}^2$ and let E be the exceptional divisor of the blow up $\pi: Y \longrightarrow X$. Let $D = \pi_*(\overline{(\pi^* L - E)})$. Then

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Indeed, $D_X = L$, so that the rhs is $\hat{\mathbb{P}}^2$, the space of lines in $\mathbb{P}^2$. But the lhs is the subspace of lines through p.

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That is, adding on E, does not make the linear system $|D|$ any larger.

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For example, the b-divisor D defined on $\mathbb{P}^2$ is not saturated with respect to the prime b-divisor E.

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- Define $D_\bullet$ by the rule

$$D_i = \frac{M_i}{i}.$$

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- $R = R(X, D)$, the flipping algebra, is exceptionally saturated.
- By Kawamata-Viehweg Vanishing, this means the restricted algebra is asymptotically A-saturated.

Shokurov algebras

Asymptotic means

$$\mathrm{Mob}\ulcorner jD_i + A\urcorner \leq jD_j.$$

for all i and j.

Definition. *Let (X, Δ) be a pair, such that $-(K_X + \Delta)$ if ample. We say that a function algebra V is a Shokurov algebra if it is bounded, asymptotically $A(X, \Delta)$-saturated and X/Z is a Fano contraction.*

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Shokurov algebras

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Theorem. *(Shokurov) Every Shokurov algebra is finitely generated, up to dimension two.*

Dimension One

By assumption $X = \mathbb{P}^1$, and we have a bounded sequence $D_\bullet$ of b-divisors, which are

$$A(X, \Delta) = -\Delta = -\sum b_m P_m = \sum a_m P_m$$

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assymptotically saturated, where $0 \leq b_m < 1$, so that $-1 < a_m \leq 0$.
As we are on a curve, we can drop the reference to b-divisors. We may write

$$D_i = \sum a_{m,i} P_m.$$

A Diophantine argument

- Asymptotic Saturation becomes:

$$\ulcorner jd_{m,i} + a_m \urcorner \leq jd_{m,j}.$$

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- (Hwk). Use Diophantine approximation to conclude that d_m is rational, and thereby finish the proof.

The surface Case

- In fact, the Diophantine approximation argument works in all dimensions, provided one can find a single model Y, on which all the b-divisors $D_\bullet$ and D are simultaneously free.

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- The surface case is especially easy, because it is not hard to show that we can take Y to be a terminal model.
- Shokurov has an appealing general conjecture, known as CCS (our first TLA), which, if true, would imply that every Shokurov algebra is finitely generated.

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Some References

- Shokurov: Prelimiting flips, Proc. Steklov Inst. of Math.v. 240, 82-219.
- Alessio Corti: 3-fold flips after Shokurov, see `http://www.dpmms.cam.ac.uk/~corti/flips_html/index.html` where there are further references.
- Florin Ambro has produced some interesting work based on Shokurov's b-divisors, see
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